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Higher Spin $N = 8$ Supergravity

E. Sezgin [1] and P. Sundell [2]

Center for Theoretical Physics, Texas A&M University,
College Station, Texas 77843, USA

ABSTRACT

The product of two $N = 8$ supersingletons yields an infinite tower of massless states of higher spin in four dimensional anti de Sitter space. All the states with spin $s \geq 1$ correspond to generators of Vasiliev's super higher spin algebra $shs^E(8|4)$ which contains the $D = 4, N = 8$ anti de Sitter superalgebra $OSp(8|4)$. Gauging the higher spin algebra and introducing a matter multiplet in a quasi-adjoint representation leads to a consistent and fully nonlinear equations of motion as shown sometime ago by Vasiliev. We show the embedding of the $N = 8$ AdS supergravity equations of motion in the full system at the linearized level and discuss the implications for the embedding of the interacting theory. We furthermore speculate that the boundary $N = 8$ singleton field theory yields the dynamics of the $N = 8$ AdS supergravity in the bulk, including all higher spin massless fields, in an unbroken phase of M-theory.

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1 Introduction

Sometime ago, the singleton representation of the super AdS group $OSp(4|8)$ was encountered in the spectrum analysis of $D = 11$ supergravity compactified on $AdS_4 \times S^7$ [1]. This is the ultra short representation of the AdS supergroup $OSp(4|8)$ in four dimensions, consisting of 8 bosonic and 8 fermionic states [2, 3], which cannot be described in terms of local fields in the bulk of AdS, as was discovered long ago by Dirac [4] in the context of the singleton representation of the AdS group $SO(3, 2)$. Indeed, it has been shown [5] that the $OSp(4|8)$ singleton multiplet occurring in the Kaluza-Klein spectrum is pure gauge. It has also been shown that, though gauge modes, the singletons are needed to fill the representations of the spectrum generating algebra $SO(8, 1)$ [6].

After the eleven dimensional supermembrane was discovered [7], it was speculated in [8] that singletons may play a role in its description. Soon after, it was conjectured in [9] and [10] that a whole class of AdS compactifications of supergravity theories may be closely related to various singleton/doubleton field theories. The field theories for them were constructed on the boundary of AdS as free superconformal field theories, but the main thrust of the conjecture remained to be tested.

Notwithstanding this state of affairs, in [11, 12, 13] the $N = 8$ supersingleton field theory was assumed to be a quantum consistent theory of the supermembrane, and its quantization was studied. In particular, the spectrum of massless states was considered. In that context, a well known and a remarkable property of the singletons was used, namely the fact that the symmetric product of two supersingletons yields an infinite tower of massless higher spin states [14]. The detailed form of this result for $OSp(8|4)$ will be spelled out in section 3. It was argued in [11, 12, 13] that all of these states should arise in the quantum supermembrane theory! Furthermore, the occurrence of the infinitely many massless higher spin fields implies the existence of infinitely many (local) gauge symmetries analogous to the Yang-Mills, general coordinate and local supersymmetries associated with spin 1, 2 and 3/2, respectively.

In a later development, in the course studying the $D = 11$ supermembranes in $AdS_4 \times S^7$, an attempt was made to obtain the singleton field theory from the $D = 11$ supermembrane action by expanding around $AdS_4 \times S^7$ [15, 16, 17]. However, for reasons explained in [16] the $N = 8$ singleton field theory on $S^2 \times S^1$ boundary of AdS_4 did not emerge fully as the critical mass term for the boson was lacking. This problem has been recently circumvented by embedding the membrane worldvolume in target space in such a way that the resulting worldvolume field theory is a conformal field theory on a three dimensional Minkowski space that serves as the boundary of AdS_4 [18].

Interestingly enough, in a related development Fradkin and Vasiliev [19, 20] were in the course of developing a higher spin gauge theory in its own right (see [19] for references to earlier work). These authors succeeded in constructing interacting field theories for higher spin fields. It was observed that the previous difficulties in constructing higher spin theories can be bypassed by formulating the theory in AdS space and to consider an infinite tower of gauge fields controlled by various higher spin algebras based on certain infinite dimensional extensions of super AdS algebras. In particular, the AdS radius could not be taken to infinity since its positive powers occurred in the higher spin interactions and therefore one could not take a naive Poincaré limit.

In a series of papers Vasiliev [22]-[29] pursued the program of constructing the AdS higher spin gauge theory and simplified the construction considerably. In [23, 24] the spin 0 and 1/2 fields were introduced to the system within the framework of free differential algebras. The theory was furthermore cast into an elegant geometrical form in [25, 26, 27, 28] by extending the higher spin algebra to include new auxiliary spinorial variables.

Remarkably, applying the formalism of Vasiliev to a suitable higher spin algebra that contains the maximally extended super AdS algebra $OSp(8|4)$, the resulting spectrum of gauge fields and spin $s \leq \frac{1}{2}$ fields coincide with the massless states resulting from the symmetric product of two $OSp(8|4$ supersingletons! [11]

With recent observation of Maldacena [31] that there is a correspondence between the physics in AdS space and at its boundary, which has been successfully tested affirmatively in a number of examples with very interesting results, the issue of whether the singletons of AdS_4 could indeed play a role in the description of bulk physics has again arisen. As far as the singleton field theory is concerned, as in the past, we still expect all the massless higher spin states to arise as two singleton states and the massive states to arise in the product of three and more singletons. An interesting question to which we have no answer at this time is how such states might arise in M-theory. However, reviving the conjecture of [11, 12, 13] we expect that they will arise as a new phase of M-theory that is yet to be uncovered.

The purpose of this paper is to address a more modest aspect of this problem by examining the Vasiliev theory of higher spin fields (which is applicable to a wide class of higher spin superalgebras) and determine the precise manner in which the $N = 8$ de Wit-Nicolai gauged supergravity [32, 33], which is the gauged version of the Cremmer-Julia $N = 8$ supergravity in four dimensions [34, 35], can be described within this framework. We will indeed show how this embedding works in the higher spin AdS supergravity based on the higher spin superalgebra known as $shs^E(8|4)$algebra [36, 37]. (The details of this algebra will be discussed in the next section). In our opinion, this constitutes a positive step towards the understanding of the M-theoretic origin of the massless higher spin gauge theory. After presenting our results, we will comment on further aspects of this issue in the concluding section.

Since the techniques for constructing higher spin theories that we shall employ in this paper are not all too well-known we have chosen to make this paper as self-contained as possible by including some basic features of the construction of higher spin theories. However, we would like to stress that the main purpose of the paper is to argue that higher spin supergravity plays a physical role within the context of M-theory and in particular that AdS bulk/boundary could be instrumental for bringing the subject of higher spin theories to a point where more nontrivial issues are addressed than previously. Therefore, while the formulation of higher spin theories allows for quite general extended algebras without any obvious truncation to supergravity, we believe that the extended higher spin theories that allow truncation to supergravity - which we shall refer to as higher spin supergravities - are of particular interest.

The organization of the paper is as follows: In section 2 we define the gauge algebra $shs^E(8|4)$ and discuss its infinite dimensional UIR's based on the $N = 8$ supersingleton. In section 3 and 4 we give the field content of the theory and the form of the nonlinear higher spin field equations. The perturbative treatment of the theory around the anti de Sitter vacuum with $N = 8$ supersymmetry is explained in section 5, where we also derive the linearized equations of motion. In section 6 we analyze the linear dynamics in more detail and in section 7 we treat the particular case of the $N = 8$ AdS supergravity. Section 8 is devoted to a further discussion of our results and some speculations. Our conventions and notation are collected in Appendices A and B. Some details of the $shs^E(8|4)$algebra are given in Appendices C and some useful lemmas are collected in Appendix D.

2 The Higher Spin Superalgebra $shs^E(8|4)$

The higher spin gauge algebra $shs^E(8|4)$ is a Lie superalgebra which we shall define as a subspace of an associative algebra $\mathcal{A}$ with Lie bracket given by the ordinary commutator based on the associative product of $\mathcal{A}$. $\mathcal{A}$ is the space of fully symmetrized functions of the two-component, complex, Grassmann even spinor elements y^α, its complex conjugate $\bar{y}^{\dot{\alpha}}$ (which together form a four-component Majorana spinor $Y_{\underline{\alpha}} = (y_\alpha, \bar{y}^{\dot{\alpha}})$ of the AdS group $SO(3,2)$; our spinor conventions are given in Appendix A) and 8 real, Grassmann odd elements θ^i forming an $SO(8)$ vector[1]. These generating elements therefore obey the "classical" commutation relations

$$
\begin{aligned}
[y_\alpha, y_\beta] &= [\bar{y}_{\dot{\alpha}}, y_\beta] = [\bar{y}_{\dot{\alpha}}, \bar{y}_{\dot{\beta}}] = 0 , \qquad \alpha, \dot{\alpha} = 1, 2 \\
[y_\alpha, \theta^i] &= [\bar{y}_{\dot{\alpha}}, \theta^i] = 0 , \\
\{\theta^i, \theta^j\} &= 0 , \qquad i = 1, ..., 8 ,
\end{aligned} \tag{1}
$$

and the reality conditions

$$
(y_\alpha)^\dagger = \bar{y}_{\dot{\alpha}} , \qquad (\theta^i)^\dagger = \theta^i . \tag{2}
$$

A general element of $\mathcal{A}$ can thus be written as a formal power series

$$
F(y, \bar{y}, \theta) = \sum_{\substack{m,n \geq 0 \\ k = 0, ..., 8}} \frac{1}{k!} F_{i_1 \cdots i_k}(m, n) \, \theta^{i_1 \cdots i_k} , \tag{3}
$$

where we have used the notation

$$
F_{i_1 \cdots i_k}(m, n) := \frac{1}{m! \, n!} F_{\alpha_1 \cdots \alpha_m \dot{\beta}_1 \cdots \dot{\beta}_n i_1 \cdots i_k} \, y^{\alpha_1 \cdots \alpha_m} \, \bar{y}^{\dot{\beta}_1 \cdots \dot{\beta}_n} ,
$$

[1]In an equivalent formulation, the Grassmann odd θ^i can be replaced by Grassmann even γ-matrices Γ^i of the $SO(8)$ Clifford algebra.

$$
\begin{aligned}
y^{\alpha_1\cdots\alpha_m} &:= y^{\alpha_1}\cdots y^{\alpha_m}\ , \qquad \bar{y}^{\dot\beta_1\cdots\dot\beta_n} := \bar{y}^{\dot\beta_1}\cdots\bar{y}^{\dot\beta_n}\ ,\\
\theta^{i_1\cdots i_k} &:= \theta^{i_1}\cdots\theta^{i_k}\ ,
\end{aligned}
\tag{4}
$$

where the coefficients $F_{\alpha_1\cdots\alpha_m\dot\beta_1\cdots\dot\beta_n i_1\cdots i_k}$ are Grassmann numbers carrying fully symmetrized spinor indices and antisymmetrized $SO(8)$ vector indices. It is important to note that the y and $\bar{y}$ dependence of (3) is not restricted by $SO(3,2)$ invariance[2]. For convenience we shall use the shorter notation

$$
F(Y,\theta\,) \ := \ F(y,\bar{y},\theta\,) \tag{5}
$$

for arbitrary elements in $\mathcal{A}$. The argument of F will be suppressed when there is no ambiguity in notation.

The associative algebra product $\star$ of $\mathcal{A}$ is defined by the following $SO(3,2)\times SO(8)$ invariant "contraction rules"[3]:

$$
\begin{aligned}
&y_{\alpha_1\cdots\alpha_m}\ \star\ y_{\beta_1\cdots\beta_n} \ = \ y_{a_1\cdots\alpha_m\beta_1\cdots\beta_n} + i\,m\,n\ y_{\alpha_1\cdots\alpha_{m-1}\beta_1\cdots\beta_{n-1}}\ \epsilon_{\alpha_m\beta_n}\\
&+i^2\,\frac{m\,(m-1)n\,(n-1)}{2!}\ y_{\alpha_1\cdots\alpha_{m-2}\beta_1\cdots\beta_{n-2}}\ \epsilon_{\alpha_{m-1}\beta_{n-1}}\ \epsilon_{\alpha_m\beta_n} + \cdots\\
&+i^{\,k}\,k!\binom{m}{k}\binom{n}{k}\ y_{\alpha_1\cdots\alpha_{m-k}\beta_1\cdots\beta_{n-k}}\ \epsilon_{\alpha_{m-k+1}\beta_{n-k+1}}\cdots\epsilon_{\alpha_m\beta_n} + \cdots\ ,
\end{aligned}
\tag{6}
$$

$$
(y_{\alpha_1\cdots\alpha_m}\bar{y}_{\dot\alpha_1\cdots\dot\alpha_n})\star\left(y_{\beta_1\cdots\beta_p}\bar{y}_{\dot\beta_1\cdots\dot\beta_q}\right) \ = \ \left(y_{\alpha_1\cdots\alpha_m}\star y_{\beta_1\cdots\beta_p}\right)\left(\bar{y}_{\dot\alpha_1\cdots\dot\alpha_n}\star\bar{y}_{\dot\beta_1\cdots\dot\beta_q}\right)\ , \tag{7}
$$

$$
\begin{aligned}
&\theta^{i_1\cdots i_m}\star\theta_{j_1\cdots j_n} \ = \ \theta^{i_1\cdots i_m}{}_{j_1\cdots j_n} + m\,n\ \theta^{i_1\cdots i_{m-1}}{}_{j_2\cdots j_n}\ \delta^{i_m}_{j_1}\\
&+\frac{m\,(m-1)n\,(n-1)}{2!}\ \theta^{i_1\cdots i_{m-2}}{}_{j_3\cdots j_n}\ \delta^{i_{m-1}i_m}_{j_2 j_1} + \cdots\\
&+k!\binom{m}{k}\binom{n}{k}\ \theta^{i_1\cdots i_{m-k}}{}_{j_{k+1}\cdots j_n}\ \delta^{i_{m-k+1}\cdots i_m}_{j_k\cdots j_1} + \cdots\qquad ,
\end{aligned}
\tag{8}
$$

where we use the following *(anti-)symmetrization convention*: all $SO(3,1)$ spinor indices of the same type are automatically subject to unit strength symmetrization, and all $SO(8)$ vector indices are automatically subject to unit strength anti-symmetrization. The $\star$ product rules (6-7) for commuting spinor elements can be summarized more concisely by the following manifestly $SO(3,2)$ invariant formula

[2] $SO(3,2)$ invariance requires a reality condition, such as $(F(y,\bar{y},\theta))^\dagger = F(y,\bar{y},\theta)$, in which case $F(y,\bar{y},\theta)$ can be expanded in terms of $SO(3,2)$ invariant monomials $F_{\underline{\alpha}_1\cdots\underline{\alpha}_m}(\theta)Y^{\underline{\alpha}_1\cdots\underline{\alpha}_m}$ where $F_{\underline{\alpha}_1\cdots\underline{\alpha}_m}(\theta)$ are real functions of θ.

[3] The $\star$ product of functions $F(y)$ is isomorphic to the algebra of fully symmetrized functions $F(\hat{y})$ of the Heisenberg algebra $[\hat{y}_\alpha,\hat{y}_\beta] = 2i\epsilon_{\alpha\beta}$, obtained from $F(y)$ by replacing $y\to\hat{y}$; see Appendix C.

$$F(Y) \star G(Y) = \int d^4U\, d^4V\, F(Y+U)\; G(Y+V)\; \exp i\left(u_\alpha v^\alpha + \bar{u}_{\dot\alpha} \bar{v}^{\dot\alpha}\right) , \tag{9}$$

where the normalization of the integration measure is such that $1 \star F = F$.

The $\star$ product rule (8) for the anti-commuting elements θ^i is equivalent to the decomposition rule for generalized Dirac matrices of the (associative) $SO(8)$ Clifford algebra

$$\theta^i \star \theta^j = \delta^{ij} + \theta^i \theta^j . \tag{10}$$

It is also straight forward to verify the associativity of (9). We also notice that as special cases of (6) we have

$$y_\alpha \star y_\beta = \eta_\alpha y_\beta + i\, \epsilon_{\alpha\beta} , \qquad \bar{y}^{\dot\alpha} \star \bar{y}^{\dot\beta} = \bar{y}^{\dot\alpha} \bar{y}^{bd} + i\, \epsilon^{\dot\alpha\dot\beta} , \tag{11}$$

which can be written in a manifestly $SO(3,2)$ covariant form

$$Y_{\underline{\alpha}} \star Y_{\underline{\beta}} = Y_{\underline{\alpha}} Y_{\underline{\beta}} + iC_{\underline{\alpha\beta}} . \tag{12}$$

The hermitian conjugation acts as an anti-linear, anti-involution of the $\star$ algebra

$$(F \star G)^\dagger = (G)^\dagger \star (F)^\dagger , \tag{13}$$

provided that we define $(\xi\eta)^\dagger = \eta^\dagger \xi^\dagger$ for Grassmann odd quantities ξ and η. In particular

$$\left(\theta^{i_1\cdots i_k}\right)^\dagger = (-1)^{\frac{k(k-1)}{2}} \theta^{i_1\cdots i_k} . \tag{14}$$

We next introduce the map

$$\tau(y_\alpha) = i\; y_\alpha , \quad \tau(\bar{y}_{\dot\alpha}) = i\; \bar{y}_{\dot\alpha} , \quad \tau(\theta^i) = i\; \theta^i . \tag{15}$$

The action of τ on a general element in $\mathcal{A}$ is then defined by declaring τ to be an involution of the classical product (1). From the definition of the $\star$ algebra in (8) and (9) it follows that τ is a graded anti-involution of $\mathcal{A}$:

$$\tau(F \star G) = (-1)^{FG}\, \tau(G) \star \tau(F) , \qquad F,\ G \in \mathcal{A} , \tag{16}$$

where the indices in the exponent indicate Grassmann parities.

We are now ready to define $shs^E(8|4)$ as the Lie superalgebra given by the subspace of $\mathcal{A}$ spanned by Grassmann *even* elements P in $\mathcal{A}$ obeying the conditions [37]

$$\tau(P) \;=\; -P\;, \qquad P^\dagger \;=\; -P\;, \tag{17}$$

and with Lie bracket defined by

$$[\,P\,,\,Q\,]_\star \;=\; P\star Q - Q\star P\;. \tag{18}$$

From (13) and (16) it follows that $shs^E(8|4)$ is closed under the bracket. The general solution to (17) is

$$\begin{aligned}
P(Y,\theta) \;=\; \frac{1}{2i}\sum_{k=0}^{\infty}\Bigg(& \sum_{m+n=4k}\left(\tfrac{1}{2!}P_{ij}(m,n)\theta^{ij}+\tfrac{1}{6!}P_{i_1\cdots i_6}(m,n)\theta^{i_1\cdots i_6}\right)\\
&+\sum_{m+n=4k+1}\left(P_i(m,n)\theta^i+\tfrac{1}{5!}P_{i_1\cdots i_5}(m,n)\theta^{i_1\cdots i_5}\right)\\
&+\sum_{m+n=4k+2}\left(P(m,n)+\tfrac{1}{4!}P_{i_1\cdots i_4}(m,n)\theta^{i_1\cdots i_4}+\tfrac{1}{8!}P_{i_1\cdots i_8}(m,n)\theta^{i_1\cdots i_8}\right)\\
&+\sum_{m+n=4k+3}\left(\tfrac{1}{3!}P_{ijk}(m,n)\theta^{ijk}+\tfrac{1}{7!}P_{i_1\cdots i_7}(m,n)\theta^{i_1\cdots i_7}\right)\Bigg)\;,
\end{aligned} \tag{19}$$

$$\left(P_{\alpha_1\cdots\alpha_m\dot\beta_1\cdots\dot\beta_n i_1\cdots i_k}\right)^\dagger \;=\; (-1)^{\frac{k\,(k-1)}{2}}\;P_{\beta_1\cdots\beta_n\dot\alpha_1\cdots\dot\alpha_m i_1\cdots i_k}\;, \tag{20}$$

where we have used the notation defined in (4). The Grassmann parity of $P_{\alpha_1\cdots\alpha_m\dot\beta_1\cdots\dot\beta_n i_1\cdots i_k}$ is $(m+n)$ mod 2. Hence the bosonic (fermionic) fields carry an even (odd) number of spinor indices and even-rank (odd-rank) antisymmetric tensor representations of $SO(8)$, that is, the 1, 28 and $70 = 35_+ + 35_-$ representations (the 8 and 56 representations) [4].

The expansion (19) is a sum of homogeneous polynomials $P_{4k+2}(Y,\theta)$ of degree $4k+2$, that is

$$P_{4k+2}(\lambda Y,\lambda\theta) \;=\; \lambda^{4k+2}P_{4k+2}(Y,\theta)\;, \qquad k=0,1,2,\ldots\;, \tag{21}$$

where λ is a complex number. Hence

$$shs^E(8|4) \;=\; \bigoplus_{k=0}^{\infty} L_k\;, \tag{22}$$

[4]The reality condition in (17) implies that the expansion (19) can be expressed in terms of $Y_{\underline{\alpha}}$ (see footnote on p. 5). For example $\sum_{m+n=4k+2}P(m,n)\to\sum_k P(4k+2)$ where $P(4k+2)$ is a $(4k+2)$'th order homogeneous polynomial in $Y_{\underline{\alpha}}$. Also note the $SO(3,2)$ invariance of the map τ, which multiplies $Y_{\underline{\alpha}}$ by a factor of i.

where the k'th "level" L_k is the vector space spanned by the polynomials P_{4k+2}. The $\star$ commutator has the following schematical structure:

$$[\,L_k\,,\,L_l\,] \;=\; \bigoplus_{m=|k-l|}^{k+l} L_m\ . \tag{23}$$

Notice that since P is Grassmann even, the $\star$ commutator of two elements in $shs^E(8|4)$ receives contributions only from odd numbers of contractions between the generators.

Eq. (23) shows that L_0 form a closed subalgebra of $shs^E(8|4)$. In fact $L_0 \simeq OSp(8|4)$ is the maximal finite dimensional subalgebra of $shs^E(8|4)$ containing $SO(3,2) \simeq Sp(4,R)$ (see Appendix B for conventions and normalizations). The $SO(8)$ subalgebra of L_0 is the diagonal subalgebra of the $SO(8)_+ \times SO(8)_-$ subalgebra of $shs^E(8|4)$ with generators $\theta^{ij} \star \frac{1}{2}(1 \pm \Gamma)$, where Γ is the hermitian and τ-invariant $SO(8)$ chirality operator

$$\Gamma \;=\; \theta^1 \cdots \theta^8\ , \qquad \{\,\Gamma\,,\,\theta^i\,\}_\star \;=\; 0\ , \qquad \Gamma^2 \;=\; 1\ , \qquad (\Gamma)^\dagger \;=\; \tau(\Gamma) \;=\; \Gamma\ . \tag{24}$$

Thus the higher spin gauge theory based on $shs^E(8|4)$ contains the gauge fields of $SO(8)_+ \times SO(8)_-$ while the truncation of the theory to the gauged $N=8$ supergravity theory contains the gauge fields of the diagonal $SO(8)$.

For $l=0$ and arbitrary k, (23) shows that L_k forms an $OSp(8|4)$ irreducible representation spanned schematically by the polynomials

$$Y^{\,4k+2}\,,\ Y^{\,4k+1}\theta\,,\ \ldots\,,\ Y^{\,4k-6}\theta^{\,8}\ , \tag{25}$$

where $Y^m\theta^n$ denotes a hermitian monomial of $(y,\bar y)$ and θ^i of homogeneous degree m and n respectively. The space spanned by $Y^m\theta^n$ forms an irreducible representations of $SO(3,2)\times SO(8)$ with $SO(3,2)$ spin $s=\frac{m}{2}$ and real dimension $\binom{m+3}{3}\binom{8}{n}$. (The spin is the eigenvalue of the $SO(3,2)$ generator M_{12}). Hence, for $k\geq 1$ the real dimension of the subspace of L_k spanned by bosonic generators is given by

$$N^b_k \;=\; \sum_{n=0}^{4} \binom{4k-3+2n}{3}\binom{8}{2n}\ , \tag{26}$$

and the real dimension of the subspace of L_k spanned by fermionic generators is given by

$$N^f_k \;=\; \sum_{n=0}^{3} \binom{4k-2+2n}{3}\binom{8}{2n+1}\ . \tag{27}$$

As expected

$$\tfrac{1}{2}\mathrm{dim}_{\mathbf{R}}(L_k) \;=\; N^b_k \;=\; N^f_k \;=\; \tfrac{256}{3}k(5+16k^2)\ , \qquad k=1,2,\ldots\ . \tag{28}$$

$k \backslash s$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4	$\cdots$	$2s$	$2s+\frac{1}{2}$	$2s+1$	$2s+\frac{3}{2}$	$2s+2$	$\cdots$
0	$35_+ + 35_-$	56	28	8	1											
1	$1+1$	8	28	56	$35_+ + 35_-$	56	28	8	1							
2					1	8	28	56	$35_+ + 35_-$	$\cdots$						
$\vdots$																
$s-1$										$\cdots$	1					
s										$\cdots$	$35_+ + 35_-$	56	28	8	1	$\cdots$
$s+1$											1	8	28	56	$35_+ + 35_-$	$\cdots$
$\vdots$																

Table 1: The $SO(3,2) \times SO(8)$ content of the symmetric tensor product of two $N=8$ singletons. This product is a unitary irreducible representation (UIR) of $shs^E(8|4)$ which decomposes into infinitely many $OSp(8|4)$ supermultiplets labeled by the level number k defined in (22). The states with $s \geq 1$ span the spectrum of the $shs^E(8|4)$ gauge fields.

The dimension grows rapidly; for example $N_1^b = N_1^f = 1,792$ and $N_2^b = N_2^f = 11,776$.

In gauging $shs^E(8|4)$ the physical states coming from the gauge fields (see section 6 for a detailed spectral analysis) are given in the $s \geq 1$ entries of Table 1 [11, 12, 36, 37]. At any level $k \geq 2$, the 512 physical states described by the gauge fields form $N=8$ supermultiplets (consisting of two irreducible $128+128$ sub-multiplets related by CPT conjugation such that the full multiplet is CPT invariant). At level $k=0,1$, however, spin $s \leq 1/2$ states are needed to form supermultiplets. The $k=1$ supermultiplet also has a total of 516 physical states, while the $k=0$ multiplet, which is CPT-conjugate, contains a total of 256 physical states.

Physical consistency of a gauge theory built on $shs^E(8|4)$ actually requires that the complete particle spectrum forms a unitary representation of the full, infinite dimensional algebra $shs^E(8|4)$ [36]; see end of section 5.1. Not all higher spin algebras are admissible in this sense. The admissibility of $shs^E(8|4)$ is based on the fact that the particle spectrum of the $shs^E(8|4)$ gauge field (that will be analyzed in detail in section 6) fits into the symmetric tensor product of two $OSp(8|4)$ singleton supermultiplets [11, 36, 37]. Each singleton supermultiplet consists of a singleton Rac in the 8_s representation of $SO(8)$ and a singleton Di in the 8_c representation:

$$Rac \oplus Di \;\; = \;\; \Big[D(\tfrac{1}{2},0) \otimes 8_s\Big] \oplus \Big[D(1,\tfrac{1}{2}) \otimes 8_c\Big] \;, \tag{29}$$

where $D(E_0,s)$ denotes an UIR of $SO(3,2)$ for which E_0 is the minimal energy eigenvalue of the energy operator M_{04}, and s is the maximum eigenvalue of the spin operator M_{12} in the lowest energy sector. From the oscillator representation of $shs^E(8|4)$ given in Appendix C it follows that the space (29) actually forms a UIR of $shs^E(8|4)$. Therefore also the symmetric and the anti-symmetric tensor products of two copies of the space (29) form UIR's of $shs^E(8|4)$. In particular, the decomposition of the symmetric tensor product

$$\Big[\Big(D(\tfrac{1}{2},0)\otimes 8_s\Big)\otimes\Big(D(\tfrac{1}{2},0)\otimes 8_s\Big)\Big]_S =$$
$$\sum_{s=0,2,4,\ldots}[D(s+1,s)\otimes 1 + D(s+1,s)\otimes 35_+] + \sum_{s=1,3,5,\ldots} D(s+1,s)\otimes 28$$
$$\Big[\Big(D(1,\tfrac{1}{2})\otimes 8_c\Big)\otimes\Big(D(1,\tfrac{1}{2})\otimes 8_c\Big)\Big]_S =$$
$$\sum_{s=0,2,4,\ldots}[D(s+1,s)\otimes 1 + D(s+1,s)\otimes 35_-] + \sum_{s=1,3,5,\ldots} D(s+1,s)\otimes 28$$
$$\Big[\Big(D(\tfrac{1}{2},0)\otimes 8_s\Big)\otimes\Big(D(1,\tfrac{1}{2})\otimes 8_c\Big)\Big]_S =$$
$$\sum_{s=\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots}(D(s+1,s)\otimes 8_v + D(s+1,s)\otimes 56_v) \tag{30}$$

under the $OSp(8|4)$ subalgebra of $shs^E(8|4)$ leads to the UIR's of $OSp(8|4)$ given in Table 1. There are a number of ways to derive this result. See, for example, [11]. Another derivation is given in Appendix C (see (267) and (268)).

We thus expect $shs^E(8|4)$ to be a suitable algebra for a supergauge theory of higher spin AdS supergravity. However, the determination of the particle content of the $shs^E(8|4)$ gauge theory requires a lot more analysis; there are auxiliary gauge fields which must be eliminated algebraically in favor of the true dynamical gauge fields and Table 1 makes it clear that one has to couple the gauge fields to a finite set of fields with spin $s \leq \frac{1}{2}$.

3 The Field Content

3.1 The $shs^E(8|4)$ Connection

The one-form master gauge connection

$$\omega\,(Y,\theta) \;=\; dx^\mu \omega_\mu(Y,\theta) \tag{31}$$

is by definition Grassmann even and takes its values in the algebra $shs^E(8|4)$:

$$\tau(\omega) \;=\; -\omega\;, \qquad (\omega)^\dagger \;=\; -\omega\;. \tag{32}$$

The curvature of ω is the $shs^E(8|4)$-valued 2-form is defined by[5]

[5]We shall temporarily set the gauge coupling equal to one. We shall discuss the gauge coupling and the gravitational coupling in section 7. Also note that in the spin $s=2$ sector one must not confuse the $SO(3,1)$

$$R = d\omega - \omega\star\omega \ , \qquad R := \tfrac{1}{2}dx^\mu\wedge dx^\nu R_{\mu\nu} \ ,$$
$$dR = \omega\star R - R\star\omega \ , \tag{33}$$

where $d = dx^\mu\partial_\mu$ is the exterior derivative and the wedge products are suppressed. The curvature transforms covariantly under $shs^E(8|4)$valued gauge transformations

$$\delta_\varepsilon\,\omega = d\,\varepsilon - [\omega\,,\,\varepsilon]_\star \ , \qquad \delta_\varepsilon\,R = [\varepsilon\,,\,R]_\star \ , \tag{34}$$

where the gauge parameter ε is an arbitrary $shs^E(8|4)$-valued function

From (19) it follows that ω has the expansion

$$\begin{aligned}\omega(Y,\theta) = \frac{1}{2i}\sum_{k=0}^{\infty}\Bigg(& \sum_{m+n=4k}\left(\tfrac{1}{2!}\omega_{ij}(m,n)\theta^{ij} + \tfrac{1}{6!}\omega_{i_1\cdots i_6}(m,n)\theta^{i_1\cdots i_6}\right)\\ + & \sum_{m+n=4k+1}\left(\omega_i(m,n)\theta^i + \tfrac{1}{5!}\omega_{i_1\cdots i_5}(m,n)\theta^{i_1\cdots i_5}\right)\\ + & \sum_{m+n=4k+2}\left(\omega(m,n) + \tfrac{1}{4!}\omega_{i_1\cdots i_4}(m,n)\theta^{i_1\cdots i_4} + \tfrac{1}{8!}\omega_{i_1\cdots i_8}(m,n)\theta^{i_1\cdots i_8}\right)\\ + & \sum_{m+n=4k+3}\left(\tfrac{1}{3!}\omega_{ijk}(m,n)\theta^{ijk} + \tfrac{1}{7!}\omega_{i_1\cdots i_7}(m,n)\theta^{i_1\cdots i_7}\right)\Bigg) \ .\end{aligned} \tag{35}$$

Note that the bosonic gauge fields are always in the 1, 28 and $35_+ + 35_-$ representations of $SO(8)$, while the fermionic fields are always in 8 and 56 representations.

By defining $\sigma^a(1,1) := (\sigma^a)_{\alpha\dot\alpha}y^\alpha\bar y^{\dot\alpha}$, $\sigma^{ab}(2,0) := \frac{1}{2}(\sigma^{ab})_{\alpha\beta}y^{\alpha\beta}$ and $\bar\sigma^{ab}(0,2) := \frac{1}{2}(\bar\sigma^{ab})_{\dot\alpha\dot\beta}\bar y^{\dot\alpha\dot\beta}$, we can decompose the gauge field $\omega_{\mu,i_1\cdots i_k}(m,n)$ and its curvature $R_{\mu\nu,i_1\cdots i_k}(m,n)$ into irreducible Lorentz tensors $\eta_{i_1\cdots i_k}(p,q)$ of spin $s=\frac{p+q}{2}$ by expanding the following $\star$ products

$$\begin{aligned}\sigma^a(1,1)\star\omega_{a,\,i_1\cdots i_k}(m,n) &:= \zeta_{i_1\cdots i_k}(m+1,n+1) + \zeta_{i_1\cdots i_k}(m+1,n-1)\\ &\quad+\zeta_{i_1\cdots i_k}(m-1,n+1) + \zeta_{i_1\cdots i_k}(m-1,n-1) \ ,\\ \sigma^{ab}(2,0)\star R_{ab,\,i_1\cdots i_k}(m,n) &:= \xi_{i_1\cdots i_k}(m+2,n) + \xi_{i_1\cdots i_k}(m,n) + \xi_{i_1\cdots i_k}(m-2,n) \ ,\\ \bar\sigma^{ab}(0,2)\star R_{ab,\,i_1\cdots i_k}(m,n) &:= \eta_{i_1\cdots i_k}(m,n+2) + \eta_{i_1\cdots i_k}(m,n) + \eta_{i_1\cdots i_k}(m,n-2) \ ,\end{aligned} \tag{36}$$

valued curvature $R_{\mu\nu,\alpha\beta}$ with the Riemann tensor $r_{\mu\nu,\alpha\beta}$. The latter is by definition the curvature of the $SO(3,1)$ gauge field $\omega_{\mu,\alpha\beta}$, while $R_{\mu\nu,\alpha\beta}$ contains contributions from $\omega\star\omega$ that are bilinear in gauge fields corresponding to generators whose commutators contain $SO(3,1)$ generators.

where the curved index μ has been converted into a flat index a by means of the vierbein $\omega_{\mu,a}$ (which is given in terms of the gauge field $\omega_{\mu,\alpha\dot{\alpha}}$ by (257)). If $m \neq n$ then these irreps are complex. If $m = n$ then $\zeta_{i_1\cdots i_k}(m+1, m+1)$ and $\zeta_{i_1\cdots i_k}(m-1, m-1)$ are real or purely imaginary (depending on k), while $\zeta_{i_1\cdots i_k}(m-1, m+1)$ and $\zeta_{i_1\cdots i_k}(m+1, m-1)$ and $\xi_{i_1\cdots i_k}(p, m)$ and $\eta_{i_1\cdots i_k}(m, p)$ ($p = m, m \pm 2$) are related by hermitian conjugation. The chiral components $\xi_{i_1\cdots i_k}(2s, 0)$ and $\eta_{i_1\cdots i_k}(0, 2s)$ ($s = 1, \frac{3}{2}, 2, ...$) are called the *generalized Weyl tensors*.

Auxiliary and Dynamical Gauge Fields and Gauge Symmetries

It is important to notice that the number of algebraically independent components of the level L_k gauge fields, as given by (28) for $k \geq 1$, grows rapidly with k, while the corresponding supermultiplets in Table 1 each contain $256+256$ states. In fact, gauge symmetry alone cannot remove all the unphysical states. One also has to impose some (curvature) constraints that serve to determine algebraically a certain subset of the gauge fields, known as *auxiliary gauge fields*, in terms of the remaining, *dynamical gauge fields*, as well as to yield dynamical field equations for the latter. (The curvature constraints also incorporate the spacetime diffeomorphisms into the gauge group). The constraints actually involve all curvature components except the generalized Weyl tensors $\xi_{i_1\cdots i_k}(2s, 0)$ and $\eta_{i_1\cdots i_k}(0, 2s)$ ($s = 1, \frac{3}{2}, 2, ...$) defined in (36). As a result only the gauge fields $\omega(m, n, \theta)$ with $|m-n| \leq 1$ are dynamical, as will be shown in section 6.1; see also Figure 1.

The auxiliary gauge fields (denoted by $\times$'s and $\diamondsuit$'s in Figure 1) are

$$\omega_a(m, n, \theta\,)\ , \qquad |m-n| \geq 2\ , \tag{37}$$

and their hermitian conjugates. Notice that there are no auxiliary gauge fields with spin $s \leq \frac{3}{2}$. In particular we shall refer to the auxiliary gauge fields $\omega_a(s-2, s, \theta\,)$ ($s = 2, 3, ...$) and their hermitian conjugates as the *generalized Lorentz connections* (and these are denoted by $\diamondsuit$'s in Figure 1).

As for the dynamical gauge fields, we categorize them as follows:

i) the *generalized vierbeins* (denoted by $\bullet$'s in Figure 1):

$$\omega_a(s-1, s-1, \theta\,)\ , \qquad s \;=\; 2, 3, 4, ...\ , \tag{38}$$

which are real,

ii) the *generalized gravitini* (denoted by $\circ$'s in Figure 1):

$$\omega_a(s-\tfrac{3}{2}, s-\tfrac{1}{2}, \theta\,)\ , \qquad s \;=\; \tfrac{3}{2}, \tfrac{5}{2}, \tfrac{7}{2}, ...\ , \tag{39}$$

and their hermitian conjugates, and

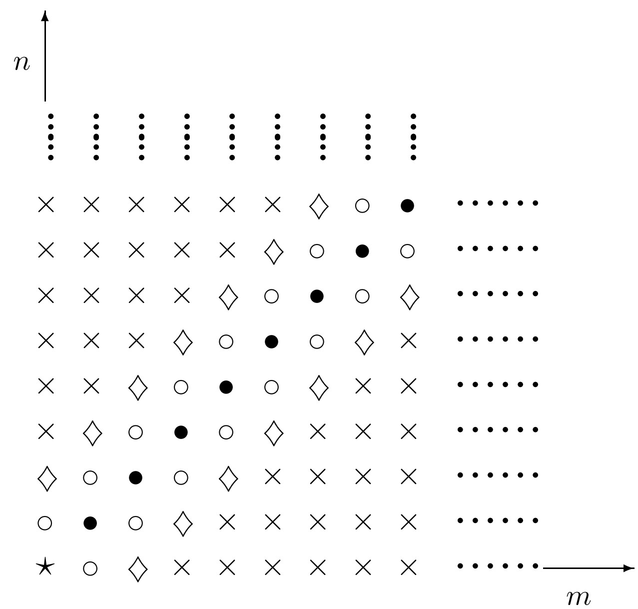

Figure 1: Each entry of the integer grid, $m, n = 0, 1, 2, ...$, represents a component $\omega(m, n; \theta\,)$ of the $shs^E(8|4)$ valued connection one form ω according to the following rule: the $\star$ denote the spin $s = 1$ component; a $\bullet$ denotes a generalized vierbein; a $\circ$ denotes a generalized gravitino; a $\diamondsuit$ denotes an auxiliary generalized Lorentz connection and the $\times$'s denote the remaining auxiliary connections.

iii) the two spin $s = 1$ $SO(8)$ gauge fields (denoted by $\star$ in Figure 1):

$$\omega_{ij}(0,0) \text{ and } \omega_{i_1 \cdots i_6}(0,0) \ . \tag{40}$$

We also differentiate between dynamical and auxiliary gauge symmetries. A *dynamical gauge symmetry* by definition has a nontrivial action on a dynamical gauge field while an *auxiliary gauge symmetry* does not act on any of the dynamical gauge fields. The dynamical gauge symmetries therefore constitute the local symmetry algebra of the dynamical equations of motion[6]. The auxiliary gauge symmetries generate algebraic shifts in the Lorentz irreps of the auxiliary gauge fields that are not determined in terms of the dynamical gauge fields by solving the constraints. The auxiliary gauge symmetries can therefore be fixed uniquely by fixing a gauge where the undetermined irreps are set equal to zero such that the auxiliary gauge fields are given uniquely in terms of the dynamical gauge fields.

The auxiliary gauge symmetries have parameters

$$\varepsilon(m, n, \theta\,) \ , \qquad |m - n| \geq 4 \tag{41}$$

[6]By definition the commutator algebra of dynamical gauge transformations closes on the dynamical gauge fields.

and hermitian conjugates. As for the dynamical gauge symmetries, in analogy with (38-40), we separate them into:

i) the *generalized Lorentz transformations* with parameters

$$\varepsilon(s-2,s,\theta\,)\ ,\qquad s=2,3,\dots\ , \tag{42}$$

and their hermitian conjugates,

ii) the *generalized reparametrizations* with parameters

$$\varepsilon(s-1,s-1,\theta\,)\ ,\qquad s=1,2,\dots\ , \tag{43}$$

which are real,

iii) the *local fermionic transformations* with parameters

$$\varepsilon(s-\tfrac{5}{2},s+\tfrac{1}{2},\theta\,)\ ,\qquad s=\tfrac{5}{2},\tfrac{7}{2},\dots\ , \tag{44}$$

and their hermitian conjugates,

iv) the *generalized local supersymmetries* with parameters

$$\varepsilon(s-\tfrac{3}{2},s-\tfrac{1}{2},\theta\,)\ ,\qquad s=\tfrac{3}{2},\tfrac{5}{2},\dots\ , \tag{45}$$

and their hermitian conjugates, and

v) the two $SO(8)$ gauge symmetries with parameters $\varepsilon_{ij}(0,0)$ and $\varepsilon_{i_1\cdots i_6}(0,0)$.

The local fermionic transformations (44) are the fermionic analogs of the generalized Lorentz transformations (42). The role of all these symmetries in arriving at the correct number of degrees of freedom will be analyzed in detail in section 6.

3.2 The Quasi-adjoint Representation

As explained at the end of section 2, the construction of a unitary gauge theory based on $shs^E(8|4)$ requires the inclusion of the spin $s\leq\frac{1}{2}$ sector shown in Table 1. To this end, one introduces a Grassmann even, zero-form master field $\phi(Y,\theta)$ [28, 27, 26, 25, 24] in the following infinite dimensional representation of $shs^E(8|4)$:

$$\tau\,(\phi\,)\ \ =\ \ \bar\pi\,(\phi\,)\ ,\qquad (\phi\,)^\dagger\ \ =\ \ \pi\,(\phi\,)\star\Gamma\ , \tag{46}$$

where Γ is the $SO(8)$ chirality operator given in (24) and π and $\bar\pi$ are involutions of the classical algebra product (1) defined by

$$\begin{array}{rclrclrcl} \pi\,(y_\alpha) & = & -y_\alpha & \pi\,(\bar{y}_{\dot\alpha}) & = & \bar{y}_{\dot\alpha} & \pi\,(\theta^i) & = & \theta^i \\ \bar\pi\,(y_\alpha) & = & y_\alpha & \bar\pi\,(\bar{y}_{\dot\alpha}) & = & -\bar{y}_{\dot\alpha} & \bar\pi\,(\theta^i) & = & \theta^i\ . \end{array} \tag{47}$$

In addition, it is useful to define the classical algebra involution

$$\pi_\theta(y_\alpha) \;=\; y_\alpha\ , \qquad \pi_\theta(\bar{y}_{\dot\alpha}) \;=\; \bar{y}_{\dot\alpha}\ , \qquad \pi_\theta(\theta^i) \;=\; -\theta^i\ . \tag{48}$$

The maps in (47-48) are also involutions of the $\star$ algebra:

$$\pi(F\star G) \;=\; \pi(F)\star\pi(G)\ , \qquad \text{idem } \bar\pi,\ \pi_\theta\ , \tag{49}$$

where F and G are arbitrary elements of $\mathcal{A}$.

The representation of $shs^E(8|4)$ on the master field ϕ is given by the gauge transformation

$$\delta_\varepsilon\,\phi \;=\; \varepsilon\star\phi-\phi\star\bar\pi\,(\varepsilon)\ ,\ , \tag{50}$$

where ε is an $shs^E(8|4)$-valued function. The covariant derivative

$$D_\omega\phi \;=\; d\phi-\omega\star\phi+\phi\star\bar\pi\,(\omega)\ . \tag{51}$$

transforms as

$$\delta_\varepsilon\,D_\omega\phi \;=\; \varepsilon\star D_\omega\phi-D_\omega\phi\star\bar\pi\,(\varepsilon) \tag{52}$$

Notice that ϕ does not quite transform in the adjoint representation due to the presence of the $\bar\pi$-operation in (50). For this reason we shall refer to the $shs^E(8|4)$ representation carried by ϕ as the *quasi-adjoint* representation[7]. The closure of $shs^E(8|4)$ on ϕ follows from (49):

$$[\,\delta_{\varepsilon_1}\,,\,\delta_{\varepsilon_2}\,]\phi \;=\; \delta_{[\,\epsilon_2\,,\,\varepsilon_1\,]_\star}\,\phi\ . \tag{53}$$

Let us emphasize that the main reason for the introduction of Γ, π and $\bar\pi$ in the definition (46) of the quasi-adjoint representation is to ensure that its spin $s\leq\frac{1}{2}$ sector correctly produces the spin $s\leq\frac{1}{2}$ states that arise from the two-singleton states tabulated in Table 1 and that must be included in the theory in order to satisfy the unitarity requirement discussed at the end of section 2. The reality condition in (46), which involves Γ in a crucial way, is engineered to be consistent with the $\tau\bar\pi$ invariance condition imposed on ϕ. This reality condition is necessary to obtain the correct field content[8].

[7] The involutions π and $\bar\pi$ are not $SO(3,2)$ invariant as opposed to the anti-involution τ. Hence, unlike ω, we cannot express ϕ in terms of a Majorana spinor of $SO(3,2)$. As a consequence, the full theory does not possess the external $SO(3,2)$ symmetry of $shs^E(8|4)$; see the discussion at the end of section 4.3.

[8] We note that the reality condition stated in (46) differs from the reality condition used in [28].

The general solution to the condition (46) is

$$\phi \;=\; C+\pi\,(C^\dagger)\star\Gamma\;, \tag{54}$$

where the field C has the expansion

$$\begin{aligned}
C(Y,\theta) \;=\; \sum_{k=0}^{\infty}\Bigg(& \sum_{\substack{m-n=4k\\ m,n\geq 0}} \Big(C(m,n)+\tfrac{1}{4!}C_{i_1\cdots i_4}(m,n)\theta^{i_1\cdots i_4}+\tfrac{1}{8!}C_{i_1\cdots i_8}(m,n)\theta^{i_1\cdots i_8}\Big)\\
&+\sum_{\substack{m-n=4k+1\\ m,n\geq 0}} \Big(\tfrac{1}{3!}C_{ijk}(m,n)\theta^{ijk}+\tfrac{1}{7!}C_{i_1\cdots i_7}(m,n)\theta^{i_1\cdots i_7}\Big)\\
&+\sum_{\substack{m-n=4k+2\\ m,n\geq 0}} \Big(\tfrac{1}{2!}C_{ij}(m,n)\theta^{ij}+\tfrac{1}{6!}C_{i_1\cdots i_6}(m,n)\theta^{i_1\cdots i_6}\Big)\\
&+\sum_{\substack{m-n=4k+3\\ m,n\geq 0}} \Big(C_{i}(m,n)\theta^{i}+\tfrac{1}{5!}C_{i_1\cdots i_5}(m,n)\theta^{i_1\cdots i_5}\Big)\Bigg)\;,
\end{aligned} \tag{55}$$

where the bosonic fields are in the 1, 28 and $35_+ + 35_-$ representations of $SO(8)$ and the fermions are in the 8 and 56 representations and the spin of $\phi_{i_1\cdots i_k}(m,n)$ is given by $s=\frac{m+n}{2}$.

We do not impose any reality condition on C. By construction C contains all solutions to $\tau(C)=\bar\pi(C)$ that have number of y spinor variables greater than or equal to the number of $\bar y$ spinor variables. Using $\tau(C^\dagger)=\pi(C)$ and $\pi\bar\pi\pi_\theta(C)=C$ (which follows from $\tau^2(C)=C$) we find that the hermitian conjugate of C obeys

$$\tau\,(C^\dagger\star\Gamma\,)\;=\;\tau\,(\Gamma\,)\star\tau\,(C^\dagger)\;=\;\Gamma\star\pi\,(C^\dagger)\;=\;\pi\,\pi_\theta(C^\dagger)\star\Gamma\;=\;\bar\pi\,(C^\dagger\star\Gamma\,)\;, \tag{56}$$

Hence, if f and g are two $\mathcal{A}$ involutions that preserve numbers of y and $\bar y$ and that commute with $\bar\pi$ and τ, then $\phi=f(C)+g(C^\dagger\star\Gamma\,)$ obeys $\tau(\phi\,)=\bar\pi(\phi\,)$. In order to solve the hermicity condition in (55) we can then take f to be the identity map and $g=\pi$:

$$\Big(C+\pi\,(C^\dagger)\star\Gamma\Big)^\dagger \;=\; \Gamma\star\,\bar\pi\,(C)+C^\dagger \;=\; \Big(\bar\pi\,\pi_\theta(C)+C^\dagger\star\Gamma\Big)\star\Gamma \;=\; \pi\;(C+\pi\,(C)\star\Gamma\,)\star\Gamma\;. \tag{57}$$

Upon substituting the expansion (55) into (54) and equating θ components we find that

$$\phi_{i_1\cdots i_k}(m,n) = \begin{cases} C_{i_1\cdots i_k}(m,n) & m>n \\ \frac{(-1)^m}{(8-k)!}\epsilon_{i_1\cdots i_k j_1\cdots j_{8-k}} C^{\star\, j_1\cdots j_{8-k}}(m,n) & m<n \end{cases} , \tag{58}$$

$$\begin{aligned} \phi(n,n) &= C(n,n) + \tfrac{1}{8!}(-1)^n\, \epsilon^{i_1\cdots i_8} C^\star_{i_1\cdots i_8}(n,n) , \\ \phi_{ijkl}(n,n) &= C_{ijkl}(n,n) + \tfrac{1}{4!}(-1)^n\, \epsilon_{ijklpqrs} C^{*\ pqrs}(n,n) , \\ \phi_{i_1\cdots i_8}(n,n) &= C_{i_1\cdots i_8}(n,n) + (-1)^n \epsilon_{i_1\cdots i_8} C^\star(n,n) , \end{aligned} \tag{59}$$

where the allowed values of m, n and k in (58) are given by (55). Thus there is an overcounting of degrees of freedom in the $m=n$ sector of C which is eliminated when C and its hermitian conjugate are added to give ϕ.

As we shall see in section 6.3, the field $\phi(0,0)$ is the $1+1$ real scalar of the level L_1 multiplet given in Table 1, and $\phi_{ijkl}(0,0)$ are the 35_++35_- real scalars of the level L_0, $N=8$ supergravity multiplet [34]. Notice that these scalars obey the reality conditions

$$\begin{aligned} \phi^\star(n,n) &= \tfrac{1}{8!}(-1)^n\, \epsilon^{i_1\cdots i_8} \phi_{i_1\cdots i_8}(n,n) , \\ \phi^*_{ijkl}(n,n) &= \tfrac{1}{4!}(-1)^n\, \epsilon_{ijklmnpq} \phi^{mnpq}(n,n) . \end{aligned} \tag{60}$$

It is gratifying to see that for $n=0$ the second equation yields the $SU(8)$ invariant reality condition on the 70 scalars of the $N=8$ supergravity [34, 35, 32, 33].

The left-handed fermions $C_\alpha^{i_1\cdots i_7}$ and their right-handed hermitian conjugates constitute the spin $s=\frac{1}{2}$ content of the $N=8$ supergravity multiplet, while the left-handed fermions C_α^{ijk} and their right-handed hermitian conjugates precisely match the spin $s=\frac{1}{2}$ content of the level $k=1$ multiplet in Table 1.

The conditions (46) imply that the quasi-adjoint representation must contain an infinite dimensional spin $s\geq 1$ sector. Thus, the inclusion of a finite number of fields with spin $s\leq\frac{1}{2}$ in the theory requires the inclusion of an infinite number of auxiliary higher spin fields. As will be shown in section 6.1, the fields $\phi_{i_1\cdots i_k}(2s+n,n)$ ($s=0,\frac{1}{2},1,...,\ n=1,2,...$) are related to the chiral components $\phi_{i_1\cdots i_k}(2s,0)$. For $s=0,\frac{1}{2}$ the chiral components are the physical fields of spin $s=0,\frac{1}{2}$. For $s\geq 1$, the $SO(8)$ content of the chiral component $\phi_{i_1\cdots i_k}(2s,0)$ matches precisely the generalized Weyl tensor $\xi_{i_1\cdots i_k}(2s,0)$ defined in (36). As will be shown in section 5.3 the linearized field equations actually give

$$\begin{aligned} \phi_{i_1\cdots i_k}(2s,0) &= \xi_{i_1\cdots i_k}(2s,0) , \\ \phi_{i_1\cdots i_k}(0,2s) &= \eta_{i_1\cdots i_k}(0,2s) , \qquad s=1,\tfrac{3}{2},2,\tfrac{5}{2},... \end{aligned} \tag{61}$$

We shall thus refer to ϕ as the *Weyl zero-form.*

4 The Higher Spin Field Equations

4.1 General Discussion

Nonlinear higher spin interactions were first constructed in [23, 24] using the formalism of free differential algebras (FDA) which aims at obtaining gauge invariant curvature constraints

$$d\omega \;=\; f^{\omega}(\omega,\phi)\;, \qquad d\phi \;=\; f^{\phi}(\omega,\phi)\;, \tag{62}$$

where the functions f^{ω} and f^{ϕ} are to be determined, order by order in ϕ, from the *integrability condition* $d^2 = 0$ and the boundary condition that (62) should reduce to the trivial constraints $R(\omega) = 0$ and $D_{\omega}\phi = 0$ in the lowest order. The structure of the gauge transformations is determined by the functions f^{ω} and f^{ϕ} and thus reduce to (34) and (50) in the lowest order. The diffeomorphism invariance of these equations is realized as gauge transformations with parameter $\epsilon = i_{\rho}\omega$, where $\delta x^{\mu} = \rho^{m}u$.

Once the explicit form of the constraints (62) has been found, the integrability guarantees that the first equation yields the one-form ω in terms of ϕ up to gauge transformations. From the second equation, one then obtains the zero-form ϕ in terms of an *initial condition* $\phi|_{p} := \phi_{p}$ where p is a fixed spacetime point. This type of initial value problem, however, is rather untractable. Instead it is more convenient to first eliminate all the auxiliary fields through the algebraic equations contained in (62), thereby obtaining a closed set of field equations involving only the spin $s \leq \frac{1}{2}$ fields and the dynamical gauge fields, and then specify the initial data and boundary conditions for these dynamical fields.

Extended Free Differential Algebra

As already mentioned, prior to deriving the dynamical field equations from (62), one first has to find the functions f^{ω} and f^{A}. This deformation problem turns out to be rather cumbersome in practice. However, there exists an elegant formalism, developed by Vasiliev [25, 26, 27, 28, 29], to facilitate the deformation procedure.

The basic idea is to generate an order by order expansion in ϕ of f^{ω} and f^{ϕ} by solving an auxiliary constraint. This constraint is formulated by means of an extended FDA with base manifold taken to be the product of ordinary spacetime with a complex space of an auxiliary spinor variable $Z_{\underline{\alpha}} = (z_{\alpha}, -\bar{z}^{\dot{\alpha}})$.

The extended FDA is of the form

$$\hat{d}A \;=\; \hat{f}^{A}(A,\Phi)\;, \qquad \hat{d}\Phi \;=\; \hat{f}^{\Phi}(A,\Phi)\;, \tag{63}$$

where $\Phi(x,Z;Y,\theta)$ is an extended Weyl zero-form, $A(x,Z;Y,\theta)$ is an extended connection one-form

$$A \;=\; W+V \;=\; dx^{\mu}W_{\mu} + dz^{\alpha}V_{\alpha} - d\bar{z}^{\dot{\alpha}}\bar{V}_{\dot{\alpha}}\;, \tag{64}$$

and $\hat{d}$ is the (x, Z) space exterior derivative

$$\hat{d} \ := \ dx^\mu \partial_\mu + dz^\alpha \partial_\alpha + d\bar{z}^{\dot{\alpha}} \bar{\partial}_{\dot{\alpha}} \ := \ d + \partial + \bar{\partial} \ := \ d + d_Z \ . \tag{65}$$

$\hat{f}^A$ and $\hat{f}^\Phi$ are given functions of A and Φ defined such that the integrability condition

$$\hat{d}^2 \ = \ 0 \tag{66}$$

is obeyed. In fact, the extended FDA describes a constraint on a Yang-Mills curvature based on an enlarged gauge algebra $\widehat{shs}^E(8|4)$ that is a Z dependent deformation of $shs^E(8|4)$ that reduces to $shs^E(8|4)$ when $Z = 0$. One also has

$$W(x, Z; Y, \theta)|_{Z=0} \ = \ \omega(Y, \theta) \ , \qquad \Phi(x, Z; Y, \theta)|_{Z=0} \ = \ \phi(Y, \theta) \ . \tag{67}$$

Before we give the details of the extended gauge theory let us first comment on the crucial features of the extension.(66) implies that (63) can be solved for A and Φ in terms of the initial condition $\Phi|_{(x,Z)=(p,0)} = \phi_p(Y, \theta)$ up to an extended gauge transformation. This shows that (63) is equivalent to a FDA of the form (62). The corresponding functions f^ω and f^ϕ are obtained by first solving for the Z dependence of W, V and Φ from the components of (63) that carries at least one Z space index. Since W and Φ are Z space zero-forms and V is a Z space one-form, this requires the initial data (67). (More precisely, the extended gauge invariance can be used to fix the gauge $V_\alpha|_{Z=0} = \bar{V}_{\dot{\alpha}}|_{Z=0} = 0$ with unbroken $shs^E(8|4)$ gauge symmetry). Denoting the solutions for W and ϕ by $W[\omega, \phi]$ and $\Phi[\phi]$ and the spacetime components of $\hat{f}^A$ and $\hat{f}^\Phi$ by f^W and f^Φ, respectively, we have

$$f^\omega(\omega, \phi) \ = \ f^W(W[\omega, \phi], \Phi[\phi])\Big|_{Z=0} \ , \qquad f^\phi(\omega, \phi) \ = \ f^\Phi(W[\omega, \phi], \Phi[\phi])\Big|_{Z=0} \ . \tag{68}$$

The fact that (68) leads to a FDA of the form (62) with a nontrivial ϕ-expansion relies on the fact that the Z space is symplectic which implies the existence of a $\star$ product of the auxiliary spinor variables. Using this $\star$ product in the definition of $\hat{f}^A$ and $\hat{f}^\Phi$ one finds

$$f^\omega(\omega, \phi) \ = \ f^W(\omega, \phi) + \dots \ , \qquad f^\phi(\omega, \phi) \ = \ f^\Phi(\omega, \phi) + \dots \ , \tag{69}$$

where $f^W(\omega, \phi)$ and $f^\Phi(\omega, \phi)$ result from the leading, classical terms in the $\star$ product of the auxiliary spinor variables, while the $\dots$ represent an expansion in ϕ coming from the higher order contractions of the auxiliary spinor variables. The latter involve derivatives of $W[\omega, \phi]$ and $\Phi[\phi]$ with respect to z_α and $\bar{z}_{\dot{\alpha}}$ which when evaluated at $Z = 0$ yield nonlinear expressions in ϕ. Hence the obtained FDA of the form (62) represents a nontrivial deformation of $R(\omega) = 0$ and $D_\omega \phi = 0$ provided we set

$$f^W(\omega, \phi) \ = \ \omega \star \omega \ , \qquad f^\phi(\omega, \phi) \ = \ \omega \star \phi - \phi \star \bar{\pi}(\omega) \ . \tag{70}$$

This corresponds to

$$\hat{f}^A = A\star A + i\,dz^2\mathcal{V}(\Phi) + i\,d\bar{z}^2(\mathcal{V}(\Phi))^\dagger \ ,$$
$$\hat{f}^\Phi = A\star\Phi - \Phi\star\bar{\pi}(A) \ , \tag{71}$$

where $\mathcal{V}$ is some function consistent with the extended integrability condition (66). Secondly, the fact that Z space is isomorphic to the space of the commuting spinors y_α and $\bar{y}_{\dot\alpha}$ allows one to define nontrivial $\star$ product contractions between the auxiliary spinor variables z^α and $z^{\dot\alpha}$, and the internal spinor variables y_α and $\bar{y}_{\dot\alpha}$. This allows one to construct a special function $\kappa(z,y)$ that projects onto anti-chiral (i.e. y independent) components, i.e. $\Phi\star\kappa|_{Z=0} = \phi|_{y=0} + \ldots$. Using κ in the definition of $\mathcal{V}$ then leads to spacetime constraints of the form (61).

4.2 Extension of The Higher Spin Superalgebra

The extended associative algebra $\widehat{\mathcal{A}}$ is by definition obtained from $\mathcal{A}$ by first extending the set of generators $(y,\bar{y},\theta)$ of $\mathcal{A}$ with the auxiliary, commuting spinors z_α and $\bar{z}_{\dot\alpha} := (z_\alpha)^\dagger$ and defining the associative, manifestly $SO(3,2)$ invariant $\star$ product on $\widehat{\mathcal{A}}$ [27, 28]

$$F(Z,Y)\ \star\ G(Z,Y) \ = \ \int F(Z+U,Y+U)\ G(Z-V,Y+V)\ \exp i\left(u_\alpha v^\alpha + \bar{u}_{\dot\alpha}\bar{v}^{\dot\alpha}\right)\ ,$$
$$[\,z_\alpha\,,\,\theta^i\,]_\star \ = \ [\,\bar{z}_{\dot\alpha}\,,\,\theta^i\,]_\star \ = \ 0\ , \tag{72}$$

where $Z_{\underline{\alpha}} := (z_\alpha, -\bar{z}^{\dot\alpha})$ is a purely imaginary Majorana spinor of $SO(3,2)$[9] and the normalization is such that $1\star F = F$. Notice that the hermitian conjugation acts as an anti-involution of $\widehat{\mathcal{A}}$. As a particular case of (72) we find (12) and

$$\begin{aligned} z_\alpha\star z_\beta &= z_\alpha z_\beta - i\,\epsilon_{\alpha\beta}\ , & \bar{z}_{\dot\alpha}\star\bar{z}_{\dot\beta} &= \bar{z}_{\dot\alpha}\bar{z}_{\dot\beta} - i\,\epsilon_{\dot\alpha\dot\beta} \\ y_\alpha\star z_\beta &= y_\alpha z_\beta - i\,\epsilon_{\alpha\beta}\ , & \bar{y}_{\dot\alpha}\star\bar{z}_{\dot\beta} &= \bar{y}_{\dot\alpha}\bar{z}_{\dot\beta} + i\,\epsilon_{\dot\alpha\dot\beta}\ , \\ z_\alpha\star y_\beta &= z_\alpha y_\beta + i\,\epsilon_{\alpha\beta}\ , & \bar{z}_{\dot\alpha}\star\bar{y}_{\dot\beta} &= \bar{z}_{\dot\alpha}\bar{y}_{\dot\beta} - i\,\epsilon_{\dot\alpha\dot\beta}\ , \end{aligned} \tag{73}$$

which can be written on the manifestly $SO(3,2)$ covariant form

$$Z_{\underline{\alpha}}\star Z_{\underline{\beta}} = Z_{\underline{\alpha}}Z_{\underline{\beta}} - iC_{\underline{\alpha\beta}}\ , \qquad Z_{\underline{\alpha}}\star Y_{\underline{\beta}} = Z_{\underline{\alpha}}Y_{\underline{\beta}} + iC_{\underline{\alpha\beta}}\ , \qquad Y_{\underline{\alpha}}\star Z_{\underline{\beta}} = Y_{\underline{\alpha}}Z_{\underline{\beta}} - iC_{\underline{\alpha\beta}}\ . \tag{74}$$

[9] Our conventions differ from [28] where the reality condition $\bar{z}_{\dot\alpha} = -z_\alpha^\dagger$ is used.

This leads to contraction rules analogous to (6) with the only difference that there is an additional factor (-1) for each contraction of type $z \star z$, $\bar{z} \star \bar{z}$, $y \star z$ and $\bar{z} \star \bar{y}$. See Appendix D for further details.

By definition the basis elements dz^α and $d\bar{z}^{\dot{\alpha}}$ of one-forms in Z-space, which form a purely imaginary, Majorana spinor $dZ^{\underline{\alpha}} = (dz^\alpha, d\bar{z}_{\dot{\alpha}})$ of $SO(3,2)$, obey

$$dZ^{\underline{\alpha}} \star F \;=\; F \star dZ^{\underline{\alpha}} \;=\; dZ^{\underline{\alpha}} F \;, \qquad (dz^\alpha)^\dagger \;=\; d\bar{z}^{\dot{\alpha}} \;, \tag{75}$$

where F is an arbitrary element of $\widehat{\mathcal{A}}$. Therefore, if we set

$$S_0 \;=\; dZ^{\underline{\alpha}} Z_{\underline{\alpha}} \;=\; dz^\alpha z_\alpha + d\bar{z}^{\dot{\alpha}} \bar{z}_{\dot{\alpha}} \;=\; (S_0)^\dagger \;, \tag{76}$$

then from the contraction rules (269) it follows that the exterior derivative d_Z in Z space can be generate by the inner, adjoint action of S_0:

$$S_0 \star F_p - (-1)^p F_p \star S_0 \;=\; -2i d_Z F_p \;, \qquad d_Z \;:=\; dz^\alpha \frac{\partial}{\partial z^\alpha} + d\bar{z}^{\,\dot{\alpha}} \frac{\partial}{\partial \bar{z}^{\dot{\alpha}}} \;, \tag{77}$$

where F_p is an $\widehat{shs}^E(8|4)$ valued (x, Z) space form of total degree p and we have defined $dz^\alpha \wedge dx^\mu = -dx^\mu \wedge dz^\alpha$. Note that (77) has the correct hermicity properties and that the associativity of the $\star$ product imply the Leibniz' rule

$$d_Z(A_p \star B_q) \;=\; d_Z A_p \star B_q + (-1)^p \, A_p \star d_Z B_q \;. \tag{78}$$

The maps τ, π and $\bar{\pi}$ defined in (15) and (47) are extended to $\widehat{\mathcal{A}}$ by setting

$$\begin{aligned}
\pi\,(z_\alpha) \;&=\; -z_\alpha \;, \qquad & \bar{\pi}\,(z_\alpha) \;&=\; z_\alpha \;, \qquad & \tau\,(z_\alpha) \;&=\; -i\, z_\alpha \;, \\
\pi\,(\bar{z}_{\dot{\alpha}}) \;&=\; \bar{z}_{\dot{\alpha}} \;, \qquad & \bar{\pi}\,(\bar{z}_{\dot{\alpha}}) \;&=\; -\bar{z}_{\dot{\alpha}} \;, \qquad & \tau\,(\bar{z}_{\dot{\alpha}}) \;&=\; -i\, \bar{z}_{\dot{\alpha}} \;,
\end{aligned} \tag{79}$$

and declaring these maps to be involutions of the extended classical algebra. We also define the action of these maps on the basis elements dz^α and $d\bar{z}^{\dot{\alpha}}$ of one-forms in Z-space by

$$d_Z(\pi(F(Z,Y,\theta)) \;=\; \pi(d_Z(F(Z,Y,\theta)) \;, \qquad \text{idem } \bar{\pi} \text{ and } \tau \;. \tag{80}$$

We next define the special $\widehat{\mathcal{A}}$ element $\kappa(z,y)$ as

$$\begin{aligned}
\kappa(z,y) \;&:=\; \exp(i\, z_\alpha y^\alpha) \;=\; \tau\,(\kappa(z,y)) \;, \\
\overline{\kappa}(\bar{z},\bar{y}) \;&:=\; (\kappa(z,y))^\dagger \;=\; \exp(-i\, \bar{z}_{\dot{\alpha}} \bar{y}^{\dot{\alpha}}) \;=\; \tau\,(\overline{\kappa}(\bar{z},\bar{y})) \;.
\end{aligned} \tag{81}$$

Using (72) we find

$$
\begin{aligned}
\kappa\star F(z,\bar z;y,\bar y;\theta) &= \kappa\, F(y,\bar z;z,\bar y;\theta)\ , & \overline{\kappa}\star F(z,\bar z;y,\bar y;\theta) &= \overline{\kappa}\, F(z,-\bar y;y,-\bar z;\theta)\ ,\\
F(z,\bar z;y,\bar y;\theta)\star\kappa &= \kappa\, F(-y,\bar z;-z,\bar y;\theta)\ , & F(z,\bar z;y,\bar y;\theta)\star\overline{\kappa} &= \overline{\kappa}\, F(z,\bar y;y,\bar z;\theta)\ ,
\end{aligned}
\tag{82}
$$

which in turn implies that the involutions π and $\bar\pi$ have inner actions in $\widehat{\mathcal{A}}$ given by

$$
\begin{aligned}
\pi(F) &= \kappa\star F\star\kappa\ , & \bar\pi(F) &= \overline{\kappa}\star F\star\overline{\kappa}\ ,\\
\kappa\star\kappa &= 1\ , & \overline{\kappa}\star\overline{\kappa} &= 1\ ,
\end{aligned}
\tag{83}
$$

where F is an arbitrary element in $\widehat{\mathcal{A}}$.

Now the enlargement $\widehat{shs}^E(8|4)$ of $shs^E(8|4)$ is defined by the extensions of (17) and (18). Thus an element $\hat P$ in $\widehat{shs}^E(8|4)$ is Grassmann even and it furthermore obeys

$$
\tau(\hat P) = -\hat P\ , \qquad (\hat P)^\dagger = -\hat P\ . \tag{84}
$$

The extension of the finite dimensional subalgebra $OSp(8|4)$ of $shs^E(8|4)$ generated by quadratic elements is given by $OSp(8|4)\times Sp(4,R)$ where the extra $Sp(4,R)$ factor is generated by the elements $z_\alpha z_\beta$, $z_\alpha \bar z_{\dot\beta}$ and $\bar z_{\dot\alpha}\bar z_{\dot\beta}$. This $Sp(4,R)$ factor commutes with $OSp(8|4)$ and generates a fictitious $Sp(4,R)$ gauge symmetry which does not arise in the spacetime FDA (62) where Z is set equal to zero.

The Extended Field Content

The extended $\widehat{shs}^E(8|4)$-valued spacetime connection one-form $W(Z,Y,\theta)=dx^\mu W_\mu$ and Weyl zero-form $\Phi(Z,Y,\theta)$ are defined by

$$
\tau(W) = -W\ , \qquad W^\dagger = -W\ , \tag{85}
$$

$$
\tau(\Phi) = \bar\pi(\Phi)\ , \qquad \Phi^\dagger = \pi(\Phi)\star\Gamma\ , \tag{86}
$$

such that the initial conditions defined in (67) indeed obey (32) and (46). Note that W and Φ are Grassmann even. The extended curvature 2-form and covariant exterior derivative

$$
\begin{aligned}
\mathcal{R} &= dW - W\star W\ ,\\
\mathcal{D}\Phi &= d\Phi - W\star\Phi + \Phi\star\bar\pi(W)\ ,
\end{aligned}
\tag{87}
$$

transform covariantly under the extended gauge transformation

$$\begin{aligned}
\delta_{\hat\varepsilon} W &= d\hat\varepsilon - [\, W\,,\,\hat\varepsilon\,]_\star\ , \\
\delta_{\hat\varepsilon}\Phi &= \hat\varepsilon \star \Phi - \Phi \star \bar\pi\,(\hat\varepsilon\,)\ ,
\end{aligned} \tag{88}$$

where the transformation parameter $\hat\varepsilon$ is an $\widehat{shs}^E(8|4)$ valued function and the variations obey the algebra $[\delta_{\hat\varepsilon_1}\delta_{\hat\varepsilon_2}] = \delta_{[\hat\varepsilon_2,\hat\varepsilon_1]_\star}$.

It is important to notice that when evaluating (87-88) at $Z=0$ there are contributions from the quadratic terms that involve $\star$ contractions of terms in W and Φ which are higher order in z and $\bar z$ (in particular there are nontrivial cross terms coming from z-y and $\bar z$-$\bar y$ contractions):

$$\begin{aligned}
\mathcal{R}|_{Z=0} &= R(\omega) + \dots\ , & \mathcal{D}\Phi|_{Z=0} &= D_\omega \phi + \dots\ , \\
\delta_{\hat\varepsilon} W|_{Z=0} &= \delta_\varepsilon \omega + \dots\ , & \delta_{\hat\varepsilon}\Phi|_{Z=0} &= \delta_\varepsilon \phi + \dots\ ,
\end{aligned} \tag{89}$$

where $\varepsilon = \hat\varepsilon|_{Z=0}$ is an $shs^E(8|4)$-valued gauge parameter and the ... represent the contributions from the higher order Taylor coefficients in the $(z,\bar z)$ expansions of W and Φ. Thus, if W and Φ were given in terms of the initial data (67), then setting $\mathcal{R}$ and $\mathcal{D}\Phi$ equal to zero would yield a nonlinear FDA of the form (62).

We proceed by defining the Z space the Grassmann even connection one-form V with components $V_{\underline{\alpha}} = (V_\alpha, \bar V^{\dot\alpha})$ introduced in (64) by

$$\tau\,(V\,) = -V\ , \qquad V^\dagger = -V\ . \tag{90}$$

We also define a Z space "covariant derivative"

$$S = S_0 + 2iV\ , \qquad \tau\,(S\,) = -S\ , \qquad S^\dagger = S\ , \tag{91}$$

where S_0 is the Z space one-form defined in (76). We take S to transform in the adjoint representation $\widehat{shs}^E(8|4)$, that is

$$\delta_{\hat\varepsilon} S = \hat\varepsilon \star S - S \star \hat\varepsilon\ . \tag{92}$$

From (77) it then follows that V indeed transforms as a Z space connection one-form

$$\delta_{\hat\varepsilon} V = d_Z \hat\varepsilon - [\, V\,,\,\hat\varepsilon\,]_\star\ . \tag{93}$$

whose curvature is related to $S \star S$ as follows

$$d_Z V - V \star V \;=\; \tfrac{1}{4} S \star S \; . \tag{94}$$

From $\tau(S_{\underline{\alpha}}) = -i\, S_{\underline{\alpha}}$ it follows that $\tau^2(S_{\underline{\alpha}}) \;=\; \pi\bar{\pi}\pi_\theta(S_{\underline{\alpha}}) \;=\; -S_{\underline{\alpha}}$. Combining with (83), we find

$$\kappa\Gamma \star S \star \kappa\Gamma \;=\; -\overline{\kappa} \star S \star \overline{\kappa} \; , \qquad \kappa\Gamma \star V \star \kappa\Gamma \;=\; -\overline{\kappa} \star V \star \overline{\kappa} \; . \tag{95}$$

4.3 The Equations of Motion in (x, Z) Space

The integrable equations of motion in (x, Z) space of the higher spin field theory are [28] [10]

$$\begin{aligned} dW &= W \star W \; , && (96)\\ d\Phi &= W \star \Phi - \Phi \star \bar{\pi}\,(W\,) \; , && (97)\\ dS &= W \star S - S \star W \; , && (98)\\ S \star \Phi &= \Phi \star \bar{\pi}\,(S\,) \; , && (99)\\ S \star S &= i\, dz^2\,(1 + \Phi \star \kappa\,\Gamma\,) + i\, d\bar{z}^2\,(1 + \Phi \star \bar{\kappa}\,) \; . && (100) \end{aligned}$$

where $dz^2 = dz^\alpha \wedge dz_\alpha$ and $d\bar{z}^2 = d\bar{z}^{\dot{\alpha}} \wedge d\bar{z}_{\dot{\alpha}} = (dz^2)^\dagger$. We shall show the integrability in section 4.4. Apart from the integrability the crucial properties of these equations are that they preserve the representation properties (85-86) and (91) and that they are invariant under the internal gauge transformations

$$\begin{aligned} \delta W &= d\hat{\varepsilon} - [\,W\,,\,\hat{\varepsilon}\,]_\star \; , \\ \delta\Phi &= \hat{\varepsilon} \star \Phi - \Phi \star \bar{\pi}\,(\hat{\varepsilon}\,) \; , \\ \delta S &= [\,\hat{\varepsilon}\,,\,S\,]_\star \; , && (101) \end{aligned}$$

where $\hat{\varepsilon}$ is an arbitrary $\widehat{shs}^E(8|4)$-valued gauge parameter. The equations are also manifestly invariant under spacetime diffeomorphisms, since they are formulated using only spacetime differential forms. In fact, the general coordinate transformation $\delta x^\mu = \rho^\mu$ is incorporated into the $\hat{\varepsilon}$-transformations by

[10] In the original formalism [27, 28] the analog of (100) is written by using κk and $\bar{\kappa}\bar{k}$, where $k, \bar{k}$ are Kleinian operator [21], instead of our $\kappa\Gamma$ and $\bar{\kappa}$, respectively. The addition of k and its hermitian conjugate $\bar{k}$ to the set of generators of the associative algebra, however, would give rise to new dynamical gauge fields which are unwanted in attempting to reproduce Table 1. The use of an $SO(2N)$ chirality operators in higher spin algebras have been discussed in [37].

$$\hat{\varepsilon}(\rho) \;=\; i_\rho W \; . \tag{102}$$

The full set of field equations (96-100) can be derived from (100) and either (97) or (98). To begin with (96) is the integrability condition for (97) and for (98). Next (99) follows from (100) by exploiting the associativity property $(S \star S) \star S = S \star (S \star S)$ and by making use of (83) and

$$\begin{aligned} dz^2 \overline{\pi}\,(S) \star \kappa\Gamma &= dz^2 \kappa\Gamma \star S \; , \\ d\bar{z}^2 \overline{\pi}\,(S) \star \overline{\kappa} &= d\bar{z}^2 \overline{\kappa} \star S \; , \end{aligned} \tag{103}$$

which in turn follow from $dz^\alpha \wedge dz^\beta \wedge dz^\gamma = 0$ and (95). Finally, (98) follows from (97) (or vice versa) by combining (97) with the covariant spacetime derivative of (100).

An Interaction Ambiguity

The extra factors of $\kappa\Gamma$ and $\overline{\kappa}$ have been inserted on the right hand side of (100) to ensure τ-invariance and gauge invariance. As already mentioned, these factors also play a crucial role in obtaining the appropriate set of constraints on the $shs^E(8|4)$ curvature $R(\omega)$. For example, the linearized contribution to $R(\omega)$ involves $\Phi \star \kappa|_{Z=0} = \phi(0, \bar{y})$ which leads to a constraint of the form (61).

These requirements do not, however, fix the right side of (100) uniquely. In order to describe the interaction ambiguity we start from the identity

$$S \star S \;=\; d\bar{Z} \left(1 + \varphi + i\Gamma^A \varphi_A \right) dZ \; , \tag{104}$$

where Γ_A $(A = 0, 1, 2, 3, 5)$ are the $SO(3,2)$ Γ-matrices, $\varphi^\dagger = \tau(\varphi) = \varphi$ is an $SO(3,2)$ scalar and $(\varphi_A)^\dagger = \tau(\varphi_A) = \varphi_A$ is an $SO(3,2)$ vector. Eq. (104) is manifestly invariant under both "internal" gauge transformations

$$\delta_{\hat{\varepsilon}}\, S \;=\; [\,\hat{\varepsilon}\,,\, S\,]_\star \; , \qquad \delta_{\hat{\varepsilon}}\, \varphi \;=\; [\,\hat{\varepsilon}\,,\, \varphi\,]_\star \; , \qquad \delta_{\hat{\varepsilon}}\, \varphi_A \;=\; [\,\hat{\varepsilon}\,,\, \varphi_A\,]_\star \tag{105}$$

and "external" $SO(3,2)$ transformations that rotate $SO(3,2)$ spinors and vectors (including $dZ^{\underline{\alpha}}$). The external transformations leave the $\star$ product (72) invariant and they cannot be incorporated into the group of internal gauge transformations since the external transformations rotate $dZ^{\underline{\alpha}}$ while the internal transformations leave $dZ^{\underline{\alpha}}$ invariant (as can be seen from (75)).

The interaction ambiguity amounts to the degrees of freedom associated with the choice of an $SO(3,1)$ invariant constraint expressing φ and φ^A as functions of the $SO(3,1)$ invariant, quasi-adjoint master field Φ.

For example, (100) is the result of the $SO(3,1)$ invariant constraint

$$\begin{aligned} \varphi &= \mathrm{Re}\,\Phi' , \qquad \varphi^5 \;=\; -\mathrm{Im}\,\Phi' , \\ \varphi^a &= 0 , \quad a = 0,1,2,3 , \end{aligned} \tag{106}$$

where Φ' obeys

$$\tau(\Phi') \;=\; \Phi' , \qquad \Phi'^\dagger \;=\; K \star \Phi' , \qquad \delta_{\hat\varepsilon}\,\Phi' \;=\; [\,\hat\varepsilon\,,\,\Phi'\,]_\star , \tag{107}$$

and $K := \kappa\bar\kappa\Gamma$, obeying $K^2 = 1$ and $\tau(K) = K^\dagger = K$. The reality condition on Φ' implies that $\mathrm{Re}\Phi' = P_+ \star \Phi'$ and $\mathrm{Im}\Phi' = -i\,P_- \star \Phi'$, where $P_\pm = \frac{1}{2}(1 \pm K)$ are projectors onto the eigenvalues ± 1 of K. This shows that the real and imaginary parts of Φ' each contain half the number of degrees of freedom of Φ'. Also notice that K commutes with W, $\hat\varepsilon$ and Φ' and anticommutes with S. Thus the result of the $SO(3,1)$ invariant constraint (106), the associativity and (98) is the following set of manifestly $SO(3,1)$ invariant equations

$$\begin{aligned} d\Phi' &= W \star \Phi' - \Phi' \star W , \\ S \star \Phi' &= \Phi' \star S , \\ S \star S &= d\bar Z \left(1 + \mathrm{Re}\Phi' + i\,\Gamma^5 \mathrm{Im}\Phi'\right) dZ . \end{aligned} \tag{108}$$

All quantities in this equation are manifestly $SO(3,2)$ invariant except the Γ^5 term. As a consequence of breaking the manifest external invariance from $SO(3,2)$ down to $SO(3,1)$, the representation of the internal $OSp(8|4)$ on the fields will only be manifestly $SO(3,1)$ invariant. The situation is analogous to the one in $N = 8$ AdS supergravity theory [30]; both equations of motion and supersymmetry transformation rules contain explicit Γ^5 matrices but in such combinations that the closure of the internal $OSp(8|4)$ algebra is not violated.

Now, eqs. (108) coincide with (97), (99) and (100) provided that we set

$$\Phi' \;=\; \Phi \star \kappa\Gamma . \tag{109}$$

The interaction ambiguity amounts to the fact this relation can be replaced by the more general one

$$\Phi' \;=\; \mathcal{V}(\Phi \star \kappa\Gamma) , \tag{110}$$

where $\mathcal{V}$ is a regular, complex $\star$ function.

4.4 Integrability of the Higher Spin Field Equations

The integrability of the higher spin field equations (96-100) in (x, Z) space can be made manifest by writing them in terms of the (x, Z) connection one-form $A = W + V$ (see (64)) and using the total exterior derivative $\hat{d} = d + d_Z$ (see (65)). One then finds that (96-100) can be cast into the form

$$F = \tfrac{i}{4}\left(dz^2\,\Phi \star \kappa\Gamma + d\bar{z}^2\,\Phi \star \overline{\kappa}\right)\ , \tag{111}$$

$$\hat{d}\Phi = A \star \Phi - \Phi \star \overline{\pi}(A)\ , \tag{112}$$

where the total curvature two-form in (x, Z) space and its Bianchi identity are given by

$$\begin{aligned} F &= \hat{d}A - A \star A\ , \\ \hat{d}F &= [\,A\,,\,F\,]_\star\ . \end{aligned} \tag{113}$$

Notice that (112) follows from inserting (111) into (113) and using (103). The gauge symmetry of (111-112) is given by

$$\delta A = \hat{d}\hat{\varepsilon} - [\,A\,,\hat{\varepsilon}\,]_\star\ , \qquad \delta\Phi = \hat{\varepsilon} \star \Phi - \Phi \star \overline{\pi}(\hat{\varepsilon})\ . \tag{114}$$

The curvature constraint (111) can be written in components as

$$\begin{aligned} F_{\mu\nu} &= F_{\alpha\mu} = F_{\dot{\alpha}\mu} = F_{\alpha\dot{\alpha}} = 0\ , \\ F_{\alpha\beta} &= -\tfrac{i}{2}\epsilon_{\alpha\beta}\Phi \star \kappa\Gamma\ , \\ F_{\dot{\alpha}\dot{\beta}} &= -\tfrac{i}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\Phi \star \overline{\kappa}\ , \end{aligned} \tag{115}$$

In order to discuss the equivalence of (111-112) and (96-100), we introduce the tri-grading (q, r, s) of $(q + r + s)$-forms in (x, Z) space where q refers to the form degree in x-space and the bi-grading (r, s) refers to the form degree in $(z, \bar{z})$-space regarded as a complex space. Thus the $(2, 0, 0)$ components of (111) yield (96). Using (77) and (91) we find that the $(1, 1, 0)$ and $(1, 0, 1)$-components yield

$$d_Z W + dV = W \star V + V \star W\ , \tag{116}$$

which is equivalent to (98). The $(0, 2, 0)$, $(0, 1, 1)$ and $(0, 0, 2)$ components yield

$$d_Z V - V \star V = \tfrac{i}{4}\,(dz^2\,\Phi \star \kappa\Gamma + d\bar{z}^2\,\Phi \star \overline{\kappa}) \tag{117}$$

which is equivalent (100) using (94). From the $(1,2,0)$-components of the Bianchi identity (113) we read off (97). Finally, the $(0,2,1)$ and $(0,1,2)$-components of (113) are equivalent to the equation

$$d_Z \Phi \;=\; V \star \Phi - \Phi \star \bar{\pi}(V) \tag{118}$$

which yields (99).

As already discussed in section 4.1, the integrability of the higher spin field equations can be used to solve for the Z dependence thus obtaining an FDA of the form (62) from which the dynamical spacetime equations follow upon the eliminating auxiliary fields. Another possibility is to begin by solving for the x dependence of W, Φ and S from (96-98) in terms of $\Phi|_p$ and $S|_p$ (where p is a fixed point in spacetime). In fact, the solution obtained is pure gauge such that the x dependence away from p is determined by an $\widehat{shs}^E(8|4)$ gauge transformation. Thus the space of gauge inequivalent solutions to the full set of equations (96-100) is equivalent to the space of gauge inequivalent solutions of the "Z space equations" obtained by inserting the pure gauge solution for W, Φ and S into the two remaining field equations (99-100):

$$\begin{aligned} S_p \star S_p &= i\, dz^2\, (1 + \Phi_p \star \kappa\, \Gamma\,) + i\, d\bar{z}^2\, (1 + \Phi_p \star \overline{\kappa}\,)\ , \\ S_p \star \Phi_p &= \Phi_p \star \bar{\pi}\,(S_p)\ . \end{aligned} \tag{119}$$

These equations are invariant under spacetime independent $\widehat{shs}^E(8|4)$ gauge transformations which are local in Z space and could be a promising starting point for obtaining other classical solutions of the theory than the AdS vacuum solution, such as solutions with nontrivial higher spin background fields or solutions that partially break the global $shs^E(8|4)$ symmetry of the AdS vacuum.

5 Expansion Around The Anti de Sitter Vacuum

5.1 The Anti de Sitter Vacuum Solution

The field equations (96-100) constitute an internally consistent set of equations, but it remains to establish the physical relevance of the equations and to make contact with $N = 8$ supergravity theory. As already discussed in section 4.1 this requires the solving of the Z space equations (98-100) and the elimination of the auxiliary fields from the algebraic equations contained in (96-97). In the following two sections we shall verify that this yields the correct free field higher spin dynamics in the AdS vacuum, and in section 7 we shall make contact with the linearized $N = 8$ supergravity model. Having established the physical consistency at the linearized level, one then has an interacting higher spin field theory based on the equations (96-100) which is tractable in classical perturbation theory.

The AdS_4 geometry can be identified as the vacuum solution given by [28]

$$\begin{aligned} \Phi_0(Z,Y,\theta) &= 0 \ , \\ S_0(Z,Y,\theta) &= dz^\alpha z_\alpha + d\bar{z}^{\dot\alpha}\bar{z}_{\dot\alpha} \ , \\ W_0(Z,Y,\theta) &= \frac{1}{4i}\Big[\ \omega_{0\,\alpha\beta}y^\alpha y^\beta + \bar\omega_{0\,\dot\alpha\dot\beta}\bar{y}^{\dot\alpha}y^{\dot\beta} + 2e_{0\,\alpha\dot\beta}y^\alpha\bar{y}^{\dot\beta}\Big] \ := \ \Omega_0(Y) \ , \end{aligned} \tag{120}$$

where the one-forms ω_0, $\bar\omega_0$ and e_0 are the vacuum Lorentz connection and vierbein of anti-de Sitter spacetime:

$$\begin{aligned} d\omega_{0\,\alpha\beta} &= \omega_{0\,\alpha\gamma}\wedge\omega_{0\,\beta}{}^\gamma + e_{0\,\alpha\dot\delta}\wedge e_{0\,\beta}{}^{\dot\delta} \ , \\ d\bar\omega_{0\,\dot\alpha\dot\beta} &= \bar\omega_{0\,\dot\alpha\dot\gamma}\wedge\bar\omega_{0\,\dot\beta}{}^{\dot\gamma} + e_{0\,\delta\dot\alpha}\wedge e_0{}^\delta{}_{\dot\beta} \ , \\ de_{0\,\alpha\dot\beta} &= \omega_{0\,\alpha\gamma}\wedge e_0{}^\gamma{}_{\dot\beta} + \bar\omega_{0\,\dot\beta\dot\delta}\wedge e_{0\,\alpha}{}^{\dot\delta} \ . \end{aligned} \tag{121}$$

To prove that (120) is a solution of the higher spin equations (96-100) one first observes that (97) and (99) are trivially satisfied while (96) reduces to (121). Using (77) one easily verifies (98) and (100).

The flat, $SO(3,2)$-valued connection one-form Ω_0 defines a AdS-covariant derivative which mixes irreducible tensors of the Lorentz $SO(3,1)$ subgroup of $SO(3,2)$ of the same spin. It splits into the curved, $SO(3,1)$-valued connection one-form

$$\omega_0 \ = \ \frac{1}{4i}\left(\ \omega_0^{\alpha\beta}y_\alpha y_\beta + \bar\omega_0^{\dot\alpha\dot\beta}\bar{y}_{\dot\alpha}y_{\dot\beta}\right) \tag{122}$$

which defines a Lorentz covariant derivative D_a which acts irreducibly on Lorentz tensors, and the vierbein

$$e_{0,\alpha\dot\alpha} \ = \ -\tfrac{1}{2}\lambda\ (\sigma_a)_{\alpha\dot\alpha}\ e_0^a \ , \qquad e_0^a \ = \ dx^\mu\, e_{0,\mu}{}^a \ , \tag{123}$$

where the mass parameter λ is given in terms of the AdS radius a by

$$\lambda \ = \ \frac{1}{a} \ . \tag{124}$$

In the following we shall need $D_{[a}D_{b]}$ acting on a Lorentz tensor $T_{a\dots\alpha\dots\dot\alpha\dots}$:

$$\begin{aligned} D_{[a}D_{b]}T_{c\dots\gamma\dots\dot\gamma\dots} \ &= \ \tfrac{1}{2}r_{ab,c}{}^d T_{d\dots\gamma\dots\dot\gamma\dots} + \ \cdots \ + \tfrac{1}{2}r_{ab,\gamma}{}^\delta T_{c\dots\delta\dots\dot\gamma\dots} + \ \cdots \\ &\quad + \tfrac{1}{2}r_{ab,\dot\gamma}{}^{\dot\delta} T_{c\dots\gamma\dots\dot\delta\dots} + \ \cdots \qquad , \end{aligned} \tag{125}$$

where the $SO(3,1)$-valued Riemann curvature

$$
\begin{aligned}
r_{ab,cd} &= -\lambda^2(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}) \ , \\
r_{ab,\gamma\delta} &= \tfrac{1}{4} r_{ab,cd}(\sigma^{cd})_{\alpha\beta} \ , \\
r_{ab,\dot\gamma\dot\delta} &= \tfrac{1}{4} r_{ab,cd}(\bar\sigma^{cd})_{\dot\alpha\dot\beta} \ .
\end{aligned} \tag{126}
$$

We shall temporarily set $\lambda = 1$ and return to the relation between λ and the dimensionful coupling of the theory in section 7.

Generalized Killing Symmetries and Admissibility Criterion for $shs^E(8|4)$

Let us emphasize that the AdS vacuum solution (120) exhibits the full $shs^E(8|4)$ symmetry and not just $OSp(8|4)$ symmetry. More explicitly, the solution (120) is invariant under gauge transformations with parameters $\hat\varepsilon_0$ obeying the following generalized Killing equations

$$
\begin{aligned}
d\hat\varepsilon_0 - [\,\Omega_0\,,\,\hat\varepsilon_0\,]_\star &= 0 \ , \\
d_Z\hat\varepsilon_0 &= 0
\end{aligned} \tag{127}
$$

whose solution space forms a superalgebra isomorphic to $shs^E(8|4)$. The last fact follows from the flatness of the $SO(3,2)$ connection Ω_0.

To construct the Killing parameters explicitly we first introduce the four commuting AdS Killing spinors $\eta^r_\alpha(x)$ $(r = 1,...,4)$ and their hermitian conjugates $\bar\eta^r_{\dot\alpha}$ obeying the AdS covariant Killing spinor equations

$$
D\eta^r_\alpha = e_{0\,\alpha\dot\beta}\bar\eta^{r\,\dot\beta} \ , \qquad D\bar\eta^r_{\dot\alpha} = e_{0\,\beta\dot\alpha}\eta^{r\,\beta} \ . \tag{128}
$$

We then define the hermitian, Grassmann even, commuting elements

$$
\eta^r = \eta^r_\alpha y^\alpha + \bar\eta^r_{\dot\alpha}\bar y^{\dot\alpha} \ , \qquad (\eta^r)^\dagger = \eta^r \ , \qquad r = 1,...,4 \ . \tag{129}
$$

From (128) it follows that the η^r obey (127). By making use of Leibniz rule it is easy to verify that

$$
\eta^{r_1\cdots r_m i_1\cdots i_n} := \eta^{(r_1}\star\cdots\star\eta^{r_m)}\theta^{i_1\cdots i_n} \ , \qquad m = 1,2,... \ , \quad k = 1,...,8 \ , \tag{130}
$$

also obey (127). Moreover, by construction

$$
\begin{aligned}
\tau(\eta^{r_1\cdots r_m i_1\cdots i_n}) &= i^{m+n}\,\eta^{r_1\cdots r_m i_1\cdots i_n}\ ,\\
\left(\eta^{r_1\cdots r_m i_1\cdots i_n}\right)^\dagger &= (-1)^{\frac{n(n-1)}{2}}\,\eta^{r_1\cdots r_m i_1\cdots i_k}\ .
\end{aligned}
\tag{131}
$$

Thus, if $\lambda_{r_1\cdots r_m i_1\cdots i_n} := \lambda_{(r_1\cdots r_m)[i_1\cdots i_n]}$ are constant Grassmann even constant coefficients obeying

$$
(\lambda_{r_1\cdots r_m i_1\cdots i_n})^\dagger = (-1)^{\frac{n(n-1)}{2}}\lambda_{r_1\cdots r_m i_1\cdots i_n}\ ,
\tag{132}
$$

then

$$
\varepsilon_0(\lambda) = i\,\lambda_{r_1\cdots r_m i_1\cdots i_n}\,\eta^{r_1\cdots r_m i_1\cdots i_n}
\tag{133}
$$

is a Killing parameter if $m+n=2$ mod 4. For given m and n, the real dimension of the space of elements of the form (133) is $\binom{m+3}{m}\binom{8}{n}$. Recalling (28) we see that for $m+n=4k+2$ ($k=0,1,...$) the space of elements of the form (133) is isomorphic to the k'th level L_k of $shs^E(8|4)$ defined in (22).

Thus the construction exhausts V_0. In particular the finite dimensional subalgebra of V_0 spanned by the 28 global $SO(8)$ parameters $\varepsilon_0(\lambda_{ij})$, the 32 unbroken AdS supersymmetries $\varepsilon_0(\lambda_{ir})$ and the 10 Killing vectors $\varepsilon_0(\lambda_{rs})$ is isomorphic to the superalgebra $OSp(8|4)$.

5.2 Perturbative Expansion

We then proceed by an order by order analysis of (96-100) by expanding them in powers of Φ which is considered to be a perturbation around the vacuum solution [28]:

$$
\begin{aligned}
\Phi &= \Phi_1+\Phi_2+\cdots\ ,\\
S &= S_0+S_1+S_2\cdots\ ,\\
W &= W_0+W_1+W_2\cdots\ ,
\end{aligned}
\tag{134}
$$

We saw that the constraint equations (99) and (100) yield (118) and (117), respectively, as a consequence of the formula (77). Using the expansion (134), we can express (118) and (117) as follows:

$$
d_Z\Phi_n = \tfrac{i}{2}\sum_{j=1}^{n-1}\left(\Phi_j\star\bar\pi\left(S_{n-j}\right)-S_j\star\Phi_{n-j}\right)\ ,
\tag{135}
$$

$$
d_Z S_n = -\tfrac12 dz^2\,\Phi_n\star\kappa\,\Gamma-\tfrac12 d\bar z^2\,\Phi_n\star\bar\kappa-\tfrac{i}{2}\sum_{j=1}^{n-1}S_j\star S_{n-j}\ ,\qquad n=1,2,...
\tag{136}
$$

This system of linear differential equations in the spinor variable Z is integrable order by order in perturbation theory. Verification of the integrability condition $d_Z^2\Phi_n = 0$ (assuming that Φ_i and S_i obey (135-136) for $i < n$) is straightforward while verification of $d_Z^2 S_n = 0$ requires the use of (103). Thus, having integrated (135-136) up to the $(n-1)$'th order one obtains the n'th order solution by first integrating (135) for Φ_n and then (136) for S_n:

$$\begin{aligned}
\Phi_n(Z,Y,\theta) &= \phi_n(Y,\theta) \\
&+\tfrac{i}{2}\sum_{j=1}^{n-1}\int_0^1 dt\,\Big\{z^\alpha\left(\Phi_j\star\overline{\kappa}\star S_\alpha^{n-j}\star\overline{\kappa}-S_\alpha^j\star\Phi_{n-j}\right)(tZ,Y,\theta) \\
&-\bar{z}^{\dot\alpha}\left(\Phi_j\star\overline{\kappa}\star\bar{S}_{\dot\alpha}^{n-j}\overline{\kappa}+\bar{S}_{\dot\alpha}^j\star\Phi_{n-j}\right)(tZ,Y,\theta)\Big\}\ , \\
S_\alpha^n(Z,Y,\theta) &= \frac{\partial}{\partial z^\alpha}\xi_n(Z,Y,\theta) \\
&+\int_0^1 dt\,t\left\{z_\alpha\Big(\Phi_n\star\kappa\Gamma+\tfrac{i}{2}\sum_{j=1}^{\left[\frac{n-1}{2}\right]}\Big[\,S_j^\beta\,,\,S_\beta^{n-j}\,\Big]_\star\Big)(tZ,Y,\theta)+\bar{z}^{\dot\alpha}\tfrac{i}{2}\sum_{j=1}^{n-1}\Big[\,S_\alpha^j\,,\,\bar{S}_{\dot\alpha}^{n-j}\,\Big]_\star(tZ,Y,\theta)\right\} \\
\bar{S}_{\dot\alpha}^n(Z,Y,\theta) &= \frac{\partial}{\partial\bar{z}^{\dot\alpha}}\xi_n(Z,Y,\theta) \\
&+\int_0^1 dt\,t\left\{\bar{z}_{\dot\alpha}\left(\Phi_n\star\overline{\kappa}+\tfrac{i}{2}\sum_{j=1}^{\left[\frac{n-1}{2}\right]}\Big[\,\bar{S}_j^{\dot\beta}\,,\,\bar{S}_{\dot\beta}^{n-j}\,\Big]_\star\right)(tZ,Y,\theta)-z^\alpha\tfrac{i}{2}\sum_{j=1}^{n-1}\Big[\,S_\alpha^j\,,\,\bar{S}_{\dot\alpha}^{n-j}\,\Big]_\star(tZ,Y,\theta)\right\}\ ,
\end{aligned} \tag{137}$$

where we have applied the integration formulae (272-274) given in Appendix D. Notice that the $\star$ products in the quadratic terms in the right hand side must be calculated with functions depending on (Z,Y,θ) before Z is replaced by tZ, as explained in Appendix D in more detail. The integration of (135) introduces the initial condition

$$\Phi_n\left(Z,Y,\theta\right)\big|_{Z=0} \;:=\; \phi_n\left(Y,\theta\right)\ ,\qquad n=1,2,...\ , \tag{138}$$

in the quasi-adjoint representation (46) of $shs^E(8|4)$. The integration of (136) introduces an exact Z-space one-form $d_Z\xi_n$, where ξ_n is an arbitrary $\widehat{shs}^E(8|4)$-valued 0-form. The perturbative expansion of the gauge transformation (92), with parameter $\hat\varepsilon=\hat\varepsilon_1+\hat\varepsilon_2+\cdots$, takes the form

$$\delta S_n \;=\; 2id_Z\hat\varepsilon_n+\sum_{j=1}^{n-1}[\,\hat\varepsilon_j\,,\,S_{n-j}\,]_\star\ ,\qquad n=1,2,... \tag{139}$$

Hence we may eliminate ξ_n by fixing the gauge

$$\xi_n \;=\; 0\;, \qquad n=1,2,... \tag{140}$$

The gauge symmetries that preserve (140) have Z-independent parameters $\varepsilon(Y,\theta)$ corresponding to the expected $shs^E(8|4)$ gauge symmetry of the field equations in x space. Notice that in the perturbative expansion (140) is imposed order by order, so that at n'th order the remaining gauge symmetries have $shs^E(8|4)$ valued parameters $\varepsilon_i(Y,\theta)$ $(i=1,..,n)$ and $\widehat{shs}^E(8|4)$ valued parameters $\hat\varepsilon_i(Z,Y,\theta)$ $(i\geq n+1)$.

Having obtained the solutions for Φ and S one proceeds by solving for W by inserting the perturbative expansions for S and W into (98). This yields the integrable set of equations

$$d_Z W_n \;=\; -\tfrac{i}{2}dS_n+\tfrac{i}{2}\sum_{j=0}^{n-1}(W_j\star S_{n-j}+S_{n-j}\star W_j)\;, \qquad n=1,2,...\;, \tag{141}$$

which allows one to obtain W_n in terms of ϕ_i $(i=1,...,n)$ and the initial conditions ω_i $(i=1,...,n)$ where

$$W_n\,(Z,Y,\theta)|_{Z=0} \;:=\; \omega_n(Y,\theta)\;, \qquad n=1,2,... \tag{142}$$

With help of the formula (272) given in Appendix D, we can integrate (141) subject to the initial condition (142), and we thus find

$$\begin{aligned} W_n(Z,Y,\theta) \;&=\; \omega_n(Y,\theta)\\ -\tfrac{i}{2}\int_0^1 dt &\left\{ z^\alpha\Big(dS^n_\alpha-\sum_{j=0}^{n-1}[\,W_j\,,\,S^{n-j}_\alpha\,]_\star\Big)(tZ,Y,\theta)+\bar z^{\dot\alpha}\Big(d\bar S^n_{\dot\alpha}-\sum_{j=0}^{n-1}[\,W_j\,,\,\bar S^{n-j}_{\dot\alpha}\,]_\star\Big)(tZ,Y,\theta)\right\}\;.\\ &n=1,2,... \end{aligned} \tag{143}$$

Having solved (98-100) and thus obtained expressions for W and Φ in terms of the initial conditions (138) and (142), one proceeds by inserting W and Φ into the remaining two equations (96-97), that is, $\mathcal{R}(Z,Y,\theta)=0$ and $\mathcal{D}\Phi(Z,Y,\theta)=0$. These equations only need to be evaluated at $Z=0$, since if (98-100) hold then $[S,\mathcal{R}]_\star=0$ and $S\star\mathcal{D}\Phi+\mathcal{D}\Phi\star\bar\pi\,(S\,)=0$ are also satisfied. The latter equations, when expanded around the AdS vacuum, yield linear Z-space differential equations which imply that $\mathcal{R}$ and $\mathcal{D}\Phi$ vanish for all Z, provided that

$$\mathcal{R}|_{Z=0} \;=\; 0\;, \qquad \mathcal{D}\Phi|_{Z=0} \;=\; 0 \tag{144}$$

are satisfied.

The initial conditions ϕ_i defined in (138) and ω_i defined in (142) are introduced independently. Since the structure of the perturbative solution is quite complicated, it seems that the perturbative solution scheme would lead to proliferation of degrees of freedom. This is misleading however, since the integrability of the exact equations implies that Φ, S and W depend only on the initial condition (67), that is on the quantities

$$\begin{aligned} \phi(Y,\theta) &:= \Phi(Z,Y,\theta)|_{Z=0} = \phi_1(Y,\theta)+\phi_2(Y,\theta)+\cdots\ , \\ \omega(Y,\theta) &:= W(Z,Y,\theta)|_{Z=0} = \Omega_0(Y)+\omega_1(Y,\theta)+\omega_2(Y,\theta)+\cdots\ . \end{aligned} \tag{145}$$

5.3 The Linearized Curvature Constraints

At the linearized level, the equations (135), (136) and (141) read

$$d_Z\Phi_1 = 0\ , \tag{146}$$

$$d_Z S_1 = -\tfrac{1}{2}\left(dz^2\Phi_1\star\kappa\,\Gamma + d\bar{z}^2\Phi_1\star\bar{\kappa}\right)\ , \tag{147}$$

$$d_Z W_1 = -\tfrac{i}{2}dS_1+\tfrac{i}{2}[\,\Omega_0,\,S_1\,]_\star\ . \tag{148}$$

From (137), (140) and (143) we find

$$\begin{aligned} \Phi_1(Z,Y,\theta) &= \phi(Y,\theta)\ , \\ S_1(Z,Y,\theta) &= \int_0^1 dt\,t\,\left(dz^\alpha z_\alpha\,\phi(-tz,\bar{y},\theta)\,\kappa(tz,y)\star\Gamma + d\bar{z}^{\dot\alpha}\bar{z}_{\dot\alpha}\,\phi(y,t\bar{z},\theta)\,\bar{\kappa}(t\bar{z},\bar{y})\right) \\ W_1(Z,Y,\theta) &= \omega(Y,\theta)+\Omega_1(Z,Y,\theta)\ , \\ \Omega_1(Z,Y,\theta) &= -\tfrac{1}{2}\int_0^1 dt'\int_0^1 dt\,t\,\Big\{\left(itt'\,{\omega_0}^{\alpha\beta}z_\alpha z_\beta + {e_0}^{\alpha\dot\beta}z_\alpha\bar{\partial}_{\dot\beta}\right)\phi(-tt'z,\bar{y},\theta)\,\kappa(tt'z,y)\star\Gamma \\ &\quad + \left(itt'{\bar{\omega}_0}^{\dot\alpha\dot\beta}\bar{z}_{\dot\alpha}\bar{z}_{\dot\beta} - {e_0}^{\alpha\dot\beta}\bar{z}_{\dot\beta}\partial_\alpha\right)\phi(y,tt'\bar{z},\theta)\,\overline{\kappa}(tt'\bar{z},\bar{y})\Big\}\ , \end{aligned} \tag{149}$$

where we have made the replacements $\phi_1\to\phi$ and $\omega_1\to\omega$ and $\partial_\alpha=\frac{\partial}{\partial y^\alpha}$. Notice that in verifying the τ invariance of (149) one has to make use of $\tau(\phi(z,\bar{y},\theta)=\pi\bar{\pi}(\phi(z,\bar{y},\theta))$. Next we insert Φ_1 and W_1 into the linearization of (144). Since ϕ has no background value the linearization of $\mathcal{D}\Phi|_{Z=0}=0$ is simply $D_{\Omega_0}\phi=0$. To linearize $\mathcal{R}|_{Z=0}=0$ it is important to notice that $\{\Omega_0,\star\Omega_1\}_\star|_{Z=0}\neq 0$, even though $\Omega_0|_{Z=0}=\Omega_1|_{Z=0}=0$. Hence the linearized higher spin equations of motion in spacetime are given by

$$
\begin{aligned}
R_1 &= \{\,\Omega_0\,,\,\Omega_1\,\}_\star|_{Z=0}\ , \\
d\phi-\Omega_0\star\phi+\phi\star\bar\pi(\Omega_0) &= 0\ ,
\end{aligned}
\tag{150}
$$

where the AdS covariant linearized curvature is defined by

$$
R_1 \ := \ d\omega-\{\,\Omega_0\,,\,\omega\,\}_\star \tag{151}
$$

which obeys the Bianchi identity

$$
dR_1 \ = \ [\,\Omega_0\,,\,R_1\,]_\star\ . \tag{152}
$$

We proceed by substituting the definition of Ω_0 given in (120) into the equations (150) and expanding all the $\star$ products. After some algebra and making use of (269) we obtain the Lorentz covariant, linearized curvature constraints

$$
\begin{aligned}
R_1(y,\bar y,\theta) &= -\tfrac{i}{4}{e_0}^{\alpha\dot\beta}\wedge {e_0}^{\gamma}{}_{\dot\beta}\ \partial_\alpha\partial_\gamma\phi(y,0,\theta)-\tfrac{i}{4}{e_0}^{\beta\dot\alpha}\wedge {e_0}_{\beta}{}^{\dot\gamma}\ \bar\partial_{\dot\alpha}\bar\partial_{\dot\gamma}\phi(0,\bar y,\theta)\star\Gamma\ , \\
D\phi(y,\bar y,\theta) &= -i\,{e_0}^{\alpha\dot\beta}\left(y_\alpha\bar y_{\dot\beta}-\partial_\alpha\bar\partial_{\dot\beta}\right)\phi(y,\bar y,\theta)\ .
\end{aligned}
\tag{153}
$$

Similarly, evaluating the $\star$ products in (151) and (152) gives the identities

$$
R_1(y,\bar y,\theta) \ := \ D\omega(y,\bar y,\theta)-{e_0}^{\alpha\dot\beta}\wedge\left(y_\alpha\bar\partial_{\dot\beta}+\bar y_{\dot\beta}\partial_\alpha\right)\omega(y,\bar y,\theta)\ , \tag{154}
$$

$$
D\,R_1(y,\bar y,\theta) \ = \ {e_0}^{\alpha\dot\beta}\wedge(y_\alpha\bar\partial_{\dot\beta}+\bar y_{\dot\beta}\partial_\alpha)R_1(y,\bar y,\theta)\ , \tag{155}
$$

where D is the Lorentz covariant, exterior derivative acting on a p-form X as

$$
DX \ = \ dX-\omega_0\star X+(-1)^pX\star\omega_0\ . \tag{156}
$$

Substituting the expansions for ω and ϕ given in (35), (54) and (55) into (153) and expanding in y and $\bar y$ gives the following component form of the linearized curvature constraints

$$
R^1_{\alpha\beta,\gamma_1\cdots\gamma_{2s-2}}(\theta\,) \ = \ C_{\alpha\beta\gamma_1\cdots\gamma_{2s-2}}(\theta\,)\ ,\qquad s=1,\tfrac{3}{2},2,...\ , \tag{157}
$$

$$
R^1_{\alpha\beta,\gamma_1\cdots\gamma_k\dot\gamma_{k+1}\cdots\dot\gamma_{2s-2}}(\theta) \ = \ 0\ ,\qquad s=\tfrac{3}{2},2,\tfrac{5}{2},...\ ,\quad k=0,..,2s-3 \tag{158}
$$

$$
\begin{aligned}
D_{\alpha\dot\alpha}C_{\beta_1\cdots\beta_m\dot\beta_1\cdots\dot\beta_n}(\theta\,) \ &= \ i\,C_{\alpha\beta_1\cdots\beta_m\dot\alpha\dot\beta_1\cdots\dot\beta_n}(\theta\,)-i\,mn\,\epsilon_{\alpha\beta_1}\,\epsilon_{\dot\alpha\dot\beta_1}\,C_{\beta_2\cdots\beta_m\dot\beta_2\cdots\dot\beta_n}(\theta\,) \\
&\qquad m,n=0,1,2,...
\end{aligned}
\tag{159}
$$

and the hermitian conjugates of (157-158). Similar manipulations of (151-152) yield

$$
\begin{aligned}
& R^1_{\alpha_1\alpha_2,\beta_1\cdots\beta_m\dot\beta_1\cdots\dot\beta_n}(\theta) \;=\; 2\,D\omega_{\alpha_1\alpha_2,\beta_1\cdots\beta_m\dot\beta_1\cdots\dot\beta_n}(\theta) \\
& -m\,\epsilon_{\alpha_1\beta_1}\,\omega_{\alpha_2}{}^{\dot\gamma}{}_{,\beta_2\cdots\beta_m\dot\gamma\dot\beta_1\cdots\dot\beta_n}(\theta) - n\,\omega_{\alpha_1\dot\beta_1,\alpha_2\beta_1\cdots\beta_m\dot\beta_2\cdots\dot\beta_n}(\theta)\ , \qquad (160) \\
& D_{\dot\alpha}{}^{\gamma}R^1_{\gamma\alpha,\beta_1\cdots\beta_m\dot\beta_1\cdots\dot\beta_n} - D_{\alpha}{}^{\dot\gamma}R^1_{\dot\gamma\dot\alpha,\beta_1\cdots\beta_m\dot\beta_1\cdots\dot\beta_n} \\
& \quad = \; m\left[R^1_{\alpha\beta_1,\beta_2\cdots\beta_m\dot\alpha\dot\beta_1\cdots\dot\beta_n} - \epsilon_{\alpha\beta_1}R^1_{\dot\alpha}{}^{\dot\gamma}{}_{,\beta_2\cdots\beta_m\dot\gamma\dot\beta_1\cdots\dot\beta_n}\right] \\
& \qquad -n\left[R^1_{\dot\alpha\dot\beta_1,\alpha\beta_1\cdots\beta_m\dot\beta_2\cdots\dot\beta_n} - \epsilon_{\dot\alpha\dot\beta_1}R^1_{\alpha}{}^{\gamma}{}_{,\gamma\beta_1\cdots\beta_m\dot\beta_2\cdots\dot\beta_n}\right]\ . \qquad (161)
\end{aligned}
$$

In writing (157-161) we have converted the curved indices of the forms into flat indices using the AdS vierbein (123) and set

$$
\begin{aligned}
D_{\alpha\dot\alpha} &:= (\sigma^a)_{\alpha\dot\alpha}D_a\ , & \omega_{\alpha\dot\beta}(Y,\theta) &:= (\sigma^a)_{\alpha\dot\beta}\,\omega_a(Y,\theta)\ , \\
R^1_{\alpha\beta}(Y,\theta) &:= \tfrac{1}{2}(\sigma^{ab})_{\alpha\beta}R^1_{ab}(Y,\theta)\ , & D\omega_{\alpha\beta}(Y,\theta) &:= \tfrac{1}{2}(\sigma^{ab})_{\alpha\beta}\,D_a\omega_b(Y,\theta)\ . \qquad (162)
\end{aligned}
$$

The linearized curvature constraints thus assume the structure suggested by the discussion at the end of section 3; the $SO(8)$ content of the generalized Weyl tensor $C_{\gamma_1\cdots\gamma_{2s}}(\theta)$ matches that of the chiral curvature $R^1_{\gamma_1\gamma_2,\gamma_3\cdots\gamma_{2s}}(\theta)$, as can be seen by comparing the expansions (35) and (55), and hence the curvature constraints (157) and (159) have well-defined θ expansions.

Independent Constraints

Not all constraints in (157-159) are independent. The relationships among them are due to the Bianchi identities (161) and the fact that some of the constraints are just trivial identifications of components in R^1 and ϕ.

To begin with, for $s \geq \frac{5}{2}$, the Bianchi identity (161) and the $k=1$ and $k=2$ components of the constraint (158) yields

$$
R^1_{\alpha\beta,\dot\alpha\dot\beta_1\cdots\dot\beta_n}(\theta) - \epsilon_{\alpha\beta}R^1_{\dot\alpha}{}^{\dot\gamma}{}_{,\dot\gamma\dot\beta_1\cdots\dot\beta_n}(\theta) \;=\; 0\ , \qquad n=2,3,\dots\ , \qquad (163)
$$

which implies that for $s\geq \frac{5}{2}$, eq. (157) simply identifies $C_{\alpha_1\cdots\alpha_{2s}}(\theta)$ with the generalized Weyl tensor of spin s. Eq. (163) also implies that the $k=0$ component of (158) is trivial.

For $s=2$, (161) and the $k=1$ component of (158) yield

$$
R^1_{\dot\alpha\dot\beta,\alpha\beta}(\theta) - \epsilon_{\dot\alpha\dot\beta}R^1_{\alpha}{}^{\gamma}{}_{,\gamma\beta}(\theta) \;=\; R^1_{\alpha\beta,\dot\alpha\dot\beta}(\theta) - \epsilon_{\alpha\beta}R^1_{\dot\alpha}{}^{\dot\gamma}{}_{,\dot\gamma\dot\beta}(\theta)\ . \qquad (164)
$$

This implies that when $R^1_{ab,\alpha\beta}(\theta)$ is expressed in terms of the vierbein by solving the $k=1$ component of (158) for $\omega_{a,\alpha\beta}(\theta)$, then $R^1_{ab}(ab,\alpha\beta)(\theta)$ consists of 20 real components: 10 components in the Weyl tensor $R_{(\alpha\beta,\gamma\delta)}(\theta)=(R_{(\dot\alpha\dot\beta,\dot\gamma\dot\delta)}(\theta))^\dagger$, 9 components in $R_{\alpha\beta,\dot\gamma\dot\delta}(\theta)=R_{\dot\gamma\dot\delta,\alpha\beta}(\theta)$ and 1 component in $R^{\alpha\beta}{}_{,\alpha\beta}(\theta)=(R^{\dot\alpha\dot\beta}{}_{,\dot\alpha\dot\beta}(\theta))^\dagger$. Hence, for $s=2$, (157) decomposes into two independent equations one of which sets the $SO(3,1)$ singlet equal to zero and the other identifying $C_{\alpha\beta\gamma\delta}(\theta)$ with the spin $s=2$ Weyl tensor. The $k=0$ component of (158) is an independent constraint that sets the 9 of $SO(3,1)$ equal to zero. In section 7 we shall show that the $9+1$ constraints contain the Ricci tensor, and they yield the linearized Einstein's equation with cosmological constant.

For $s=\frac{3}{2}$, eq. (157) decomposes into two independent equations one of which is $R^{1\,\beta}_{\alpha\ \ ,\beta}(\theta)=0$ and the other identifying $C_{\alpha\beta\gamma}(\theta)$ with the spin $s=\frac{3}{2}$ Weyl tensor $R^1_{(\alpha\beta,\gamma)}(\theta)$. Eq. (158), where only $k=0$ is allowed, is the independent constraint $R^1_{\alpha\beta,\dot\gamma}(\theta)=0$. In section 7 we shall show that the independent constraints in this sector contain the gravitino field equation.

For $s=1$ (157) identifies $C_{\alpha\beta}(\theta)$ with the $SO(8)\times SO(8)$ field strength $R^1_{\alpha\beta}(\theta)$ and likewise, for $(m,n)=(0,0)$ (159) identifies $C_{\alpha\dot\alpha}(\theta)$ with the derivatives of the scalars.

Turning to (159), we note that the Bianchi identity (161) and the identifications in (157) yield

$$D_{\dot\alpha}{}^{\beta}C_{\beta\alpha_1\cdots\alpha_l}(\theta)\ =\ 0\ ,\qquad l=2,3,\dots\ , \tag{165}$$

which implies that (159) is simply an identification of $C_{\alpha\beta_1\cdots\beta_m\dot\alpha}(\theta)$ with $D_{\dot\alpha(\alpha}C_{\beta_1\cdots\beta_m)}(\theta)$ for $m\geq 3$. On the other hand, for $(m,n)=(2,0)$, $(1,0)$, $(1,1)$ it is clear that (159) yields independent constraints, leading to the spin $s\leq 1$ equations of motion. The remaining components of (159) (that is for $m+n\geq 3$ and $m,n\geq 1$) are redundant. To show this we assume that (159) holds for $D_{\alpha\dot\alpha}C_{\beta_1\cdots\beta_{m-1}\dot\beta_1\cdots\dot\beta_{n-1}}(\theta)$. Then it follows that

$$\begin{aligned} D_{\beta_1}{}^{\dot\alpha}C_{\beta_2\cdots\beta_{m+1}\dot\alpha\dot\beta_1\cdots\dot\beta_{n-1}}(\theta) &= 0\ ,\\ D^{\alpha\dot\alpha}C_{\alpha\beta_1\cdots\beta_{m-1}\dot\alpha\dot\beta_1\cdots\dot\beta_{n-1}}(\theta) &= -i\,(m+1)(n+1)C_{\beta1m-1\dot\beta_1\cdots\dot\beta_{n-1}}(\theta)\ ,\end{aligned} \tag{166}$$

which in turn implies that (159) holds for $D_{\alpha\dot\alpha}C_{\beta_1\cdots\beta_m\dot\beta_1\cdots\dot\beta_n}(\theta)$.

To summarize, the independent constraints are

(i) the $k\geq 1$ components of (158) for $s\geq 2$:

$$\begin{aligned} R^1_{\alpha\beta,\gamma_1\cdots\gamma_k\dot\gamma_{k+1}\cdots\dot\gamma_{2s-2}}(\theta) &= 0\ ,\\ R^1_{\dot\alpha\dot\beta,\gamma_1\cdots\gamma_k\dot\gamma_{k+1}\cdots\dot\gamma_{2s-2}}(\theta) &= 0\ ,\qquad s=2,\tfrac{5}{2},3,\dots\ ,\qquad k=1,\dots,2s-3\ ,\end{aligned} \tag{167}$$

(ii) the following components of (157) and (158) for $s=2$:

$$
\begin{aligned}
R^1_{\alpha\beta,}{}^{\alpha\beta}(\theta\,) &= 0\ , \\
R^1_{\alpha\beta,\dot\gamma\dot\delta}(\theta\,) &= 0\ ,
\end{aligned}
\tag{168}
$$

(iii) the following components of (157) and (158) for $s=\frac{3}{2}$:

$$
\begin{aligned}
R^1_{\alpha\beta,}{}^{\beta}(\theta\,) &= R^1_{\dot\alpha\beta,}{}^{\beta}(\theta\,) = 0\ , \\
R^1_{\alpha\beta,\dot\gamma}(\theta\,) &= R^1_{\dot\alpha\dot\beta,\gamma}(\theta\,) = 0\ ,
\end{aligned}
\tag{169}
$$

(iv) and the following components of (159) for $s=0,\frac{1}{2},1$:

$$
\begin{aligned}
D_{\alpha\dot\alpha}C_{\beta\gamma}(\theta\,) &= i\,C_{\alpha\beta\gamma\dot\alpha}(\theta\,)\ , \\
D_{\alpha\dot\alpha}C_{\beta}(\theta\,) &= i\,C_{\alpha\beta\dot\alpha}\ , \\
D_{\alpha\dot\alpha}\phi_{\beta\dot\beta}(\theta\,) &= i\,\phi_{\alpha\dot\alpha}(\theta\,)-i\,\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\,\phi(\theta\,)\ ,
\end{aligned}
\tag{170}
$$

together with the identities

$$
C_{\alpha\beta}(\theta) = R^1_{\alpha\beta}(\theta)\ ,\qquad \phi_{\alpha\dot\alpha}(\theta) = -i\,D_{\alpha\dot\alpha}\phi(\theta)\ . \tag{171}
$$

The above analysis shows the auxiliary status of the zero-forms $\phi(m,n,\theta)$ with $m+n\geq 2$ and the dynamical status of the spin $s\leq\frac{1}{2}$ fields $\phi(m,n,\theta)$ with $m+n\leq 1$.

Symmetries of the Linearized Equations

Inserting the expansion (134), and similar expansions for the gauge parameters where the leading term is the Killing parameter, into the expression (101) for the gauge transformations, linearizing and setting $Z=0$ using (89) we find that the linearized field equations (157-159) are invariant under the Killing symmetries

$$
\delta_{\varepsilon_0}\omega = [\,\varepsilon_0\,,\,W_1\,]_\star|_{Z=0}\ ,\qquad \delta_{\varepsilon_0}\phi = \varepsilon_0\star\phi-\phi\star\bar\pi\,(\varepsilon_0\,)\ , \tag{172}
$$

where ε_0 obey (127), and the local gauge transformations

$$
\delta_\varepsilon\omega = d\varepsilon-[\,\Omega_0\,,\,\varepsilon\,]_\star\ ,\qquad \delta_\varepsilon\phi = 0\ , \tag{173}
$$

where ε is an arbitrary $shs^E(8|4)$-valued parameter. These transformation in components read

$$\delta\omega_{\alpha\dot\alpha,\beta_1\cdots\beta_m\dot\beta_1\cdots\dot\beta_n} \;=\; D_{\alpha\dot\alpha}\varepsilon_{\beta_1\cdots\beta_m\dot\beta_1\cdots\dot\beta_n} + m\,\epsilon_{\alpha\beta_1}\varepsilon_{\beta_2\cdots\beta_m\dot\beta_1\cdots\dot\beta_n} + n\,\epsilon_{\dot\alpha\dot\beta_1}\varepsilon_{\beta_1\cdots\beta_m\dot\beta_2\cdots\dot\beta_n}\ . \tag{174}$$

The linearized field equations transform into each other under (172) but they are separately invariant under (173). The local symmetries (173) will be used to impose gauge conditions on the gauge field ω.

6 Spectral Analysis

In section 6.1 we shall analyze the elimination of the auxiliary gauge fields $\omega(m,n,\theta)$ with $|m-n|>1$ by solving (167). In section 6.2 we shall analyze the equations of motion in the $s\geq\frac{3}{2}$ sector of the theory. For $s=\frac{3}{2}$ and $s=2$, these follow from (168) and (169), respectively, and for $s\geq\frac{5}{2}$ they follow from (167). Finally, in section 6.3 we shall analyze the equations of motion in the $s\leq 1$ sector following from (170).

For $s=\frac{5}{2},\frac{7}{2},...$, the dynamical field equations are "Dirac like" first order equations obtained from some of the Lorentz irreps of the physical fermionic curvatures $R_1(s-\frac{3}{2},s-\frac{1}{2},\theta)$ (the ones that are not used up in solving for the auxiliary gauge fields $\omega(s-\frac{5}{2},s+\frac{1}{2},\theta)$). For $s=3,4,...$ the dynamical field equations are the "Klein-Gordon like" second order equations obtained from some of the Lorentz irreps of the auxiliary bosonic curvatures $R_1(s-2,s,\theta)$ (the ones that are not used up in solving for $\omega(s-3,s+1,\theta)$) [11].

6.1 Elimination of Auxiliary Fields

The strategy for solving the generalized torsion constraints (167) is to first decompose the gauge fields and their derivatives and the gauge transformations (174) into Lorentz irreducible tensors. An auxiliary gauge field is then a gauge field that can be eliminated completely in the sense that each irrep of the gauge field can either be solved for algebraically using (167) or be set equal to zero by using the gauge transformations that take the form of Stuckelberg type shifts.

In order to solve the constraints (167-169), we first solve (167) which can be written as $R^1_{ab}(m,n)=0$, [12] for $|m-n|=0,1,2$ and the constraints (168-169). Using these results, we then solve the constraints (167) for the remaining values $|m-n|\geq 3$.

[11] In the action formalism the gauge invariant quadratic action leading to the correct dynamical equations of motion fails in producing the constraints allowing to solve for $\omega(m,n,\theta)$ with $|m-n|>2$. These therefore have to imposed by hand, since it is crucial to solve for all auxiliary fields at the linearized level in order for higher order interactions to make sense [22]. Thus, the description of the interacting higher spin gauge theory seems to be more cumbersome in the action formalism than in the free differential algebra approach where the field equations are obtained from an integrable set of constraints.

[12] In this and the next section we will suppress the θ-dependence of all the fields and parameters.

The $m-n=0,\ s\geq 2$ Sector:

For convenience let us write $m=n=s-1$. From the decomposition rules (36), we find that the constraint $R^1_{ab}(s-1,s-1)=0$ has the schematical structure (suppressing the exact s-dependent values of the coefficients)

$$
\begin{aligned}
\lambda(s+1,s-1)+\bar{\zeta}(s+1,s-1) &= 0\ ,\\
\lambda(s-1,s-1)+\zeta(s-1,s-1)+\bar{\zeta}(s-1,s-1) &= 0\ ,\\
\lambda(s-3,s-1)+\zeta(s-3,s-1) &= 0\ ,
\end{aligned}
\tag{175}
$$

where λ's are the three irreps that appear in the decomposition of the self-dual part of $D_{[a}\omega_{b]}(s-1,s-1)$ and ζ and $\bar{\zeta}$ are irreps that appear in the decomposition of the generalized Lorentz connection $\omega_{\alpha\dot{\alpha}}(s-2,s)$ and its hermitian conjugate $\omega_{\alpha\dot{\alpha}}(s,s-2)$. Thus we can solve for $\eta(s-1,s+1)$ in terms of $\lambda(s+1,s-1)^\dagger$, and $\zeta(s-1,s-1)$ in terms of $\lambda(s-1,s-1)$ and $\lambda(s-1,s-1)^\dagger$, and $\zeta(s-3,s-1)$ in terms of $\lambda(s-3,s-1)$. The irrep $\zeta(s-3,s+1)$ for $s\geq 3$ does not appear in (175) and therefore it remains undetermined. Putting these results together, we find the following relation between the generalized Lorentz connections and generalized vierbeins:

$$
\begin{aligned}
\omega_{\alpha\dot{\alpha},\beta_1\cdots\beta_{s-2}\dot{\beta}_1\cdots\dot{\beta}_s} &= -\tfrac{s-3}{s-1}D\omega_{\dot{\alpha}\dot{\beta}_1,\alpha\beta_1\cdots\beta_{s-2}\dot{\beta}_2\cdots\dot{\beta}_s}+D\omega_{\dot{\beta}_1\dot{\beta}_2,\alpha\beta_1\cdots\beta_{s-2}\dot{\alpha}\dot{\beta}_3\cdots\dot{\beta}_s}\\
&\quad+\tfrac{3}{s+1}\epsilon_{\dot{\alpha}\dot{\beta}_1}D\omega_{\alpha}{}^{\gamma}{}_{,\gamma\beta_1\cdots\beta_{s-2}\dot{\beta}_2\cdots\dot{\beta}_s}+\tfrac{s-2}{s+1}\epsilon_{\dot{\alpha}\dot{\beta}_1}D\omega_{\beta_1}{}^{\gamma}{}_{,\gamma\alpha\beta_2\cdots\beta_{s-2}\dot{\beta}_2\cdots\dot{\beta}_s}\\
&\quad+(s-2)\epsilon_{\alpha\beta_1}\zeta_{\beta_2\cdots\beta_{s-2}\dot{\alpha}\dot{\beta}_1\cdots\dot{\beta}_s}\ ,\qquad s=2,3,4,\ldots
\end{aligned}
\tag{176}
$$

Another way of understanding the presence of the undetermined irrep $\zeta(s-3,s+1)$ in (176) for $s\geq 3$ is to note that the gauge symmetry with parameter $\varepsilon(s-3,s+1)$ transforms the generalized Lorentz connection but not the generalized vierbein; from (174) and (176) it follows that $\delta\zeta(s-3,s+1)=\varepsilon(s-3,s+1)$ $(s\geq 3)$. Since this gauge symmetry acts by shifting $\zeta(s-3,s+1)$ it can be fixed uniquely by imposing the gauge condition

$$
\zeta(s-3,s+1) = 0\ ,\qquad s=3,4,\ldots\ .
\tag{177}
$$

Hence the generalized Lorentz connections and the gauge symmetry with parameter $\varepsilon(s-3,s+1)$ are auxiliary.

There are also constraints which arise from the Bianchi identities (161) upon the use of the use of the vanishing curvature constraints (167). Using the decomposition rules (36) and using the constraints $R^1_{ab}(s-1,s-1)=0$ in the Bianchi identity (161) for $s\geq 2$, we obtain

$$
\xi(s,s) = \bar{\eta}(s,s)\ ,
$$

$$
\begin{aligned}
\eta(s-2,s-2) &= \bar{\xi}(s-2,s-2)\ , \\
(s-2)\xi(s-2,s)+\eta(s-2,s) &= 0\ , \qquad s=2,3,\dots\ ,
\end{aligned}
\tag{178}
$$

where (ξ,η) and $(\bar{\xi},\bar{y})$ are the irreps that appear in the decompositions of $R^1_{ab}(s-2,s)$ and $R^1_{ab}(s,s-2)$, respectively. The first two equations state that $\xi(s,s)$ and $\eta(s-2,s-2)$ are real.

The $|m-n|=2$, $s\geq 2$ Sector:

For convenience, we set $m=s-2$ and $n=s$. Taking into account (178) in the constraint $R^1_{ab}(s-2,s)=0$, we find

$$
\xi(s,s) = \eta(s-2,s-2) = 0\ , \qquad s=2,3,\dots\ ,
\tag{179}
$$

and

$$
\begin{aligned}
\xi(s-2,s)+\zeta^{(-)}(s-2,s) &= 0 \\
\xi(s-4,s) = \eta(s-2,s+2) &= 0\ , \qquad s=3,4,\dots
\end{aligned}
\tag{180}
$$

None of the irreps in (180), nor any other of the remaining vanishing curvatures components, depend on the generalized vierbein. By examining the exact coefficients of the last equation in (178) and the first equation in (180) one can show that these two equations are linearly independent and that the latter one is independent of the generalized vierbein. Moreover the generalized vierbein does not appear undifferentiated in any of the remaining curvature constraints. Hence the generalized vierbeins are dynamical and obey the equations of motion (179), which can be written more explicitly as

$$
e_{0,\alpha_1}{}^{\dot{\beta}}\wedge R^1_{\alpha_2\cdots\alpha_{s-1}\dot{\beta}\dot{\alpha}_1\cdots\dot{\alpha}_{s-1}} = 0\ , \qquad s=2,3,\dots
\tag{181}
$$

The $|m-n|=1$, $s\geq\frac{3}{2}$ Sector

For convenience, we set $m=s-\frac{3}{2}$ and $n=s-\frac{1}{2}$. The constraint $R^1_{ab}(s-\frac{3}{2},s-\frac{1}{2})=0$ for $s\geq\frac{5}{2}$, together with (169) and the Bianchi identity (161) imply

$$
\begin{aligned}
\xi(s+\tfrac{1}{2},s-\tfrac{1}{2}) &= 0\ , \\
(s-\tfrac{3}{2})\zeta^{(-)}(s-\tfrac{3}{2},s-\tfrac{5}{2}) &= 0\ , \\
(s-\tfrac{3}{2})\xi(s-\tfrac{3}{2},s-\tfrac{1}{2})+\eta(s-\tfrac{3}{2},s-\tfrac{1}{2}) &= 0\ , \qquad s=\tfrac{3}{2},\tfrac{5}{2},\dots
\end{aligned}
\tag{182}
$$

and

$$\begin{aligned}\xi(s-\tfrac{3}{2},s-\tfrac{1}{2})+\eta(s-\tfrac{3}{2},s-\tfrac{1}{2}) &= 0\\ (s-\tfrac{5}{2})\xi(s-\tfrac{7}{2},s-\tfrac{1}{2}) &= \zeta^{(-)}(s-\tfrac{3}{2},s+\tfrac{3}{2}) = 0\ , \qquad s=\tfrac{5}{2},\tfrac{7}{2},...\ .\end{aligned} \tag{183}$$

Following steps analogous to those used in the analysis of (179-180) we find that none of the irreps in (183), nor any other of the remaining vanishing curvatures components, depend on the generalized gravitini. The two linear combinations of $\xi(s-\frac{3}{2},s-\frac{1}{2})$ and $\eta(s-\frac{3}{2},s-\frac{1}{2})$ that appear in (182) and (183) have different coefficients; the linear combination in (183) is independent of the generalized gravitino. Hence the generalized gravitini are dynamical and obey the equations of motion (182), which can be written more explicitly as

$$e_{0,\alpha_1}{}^{\dot\beta}\wedge R^1_{\alpha_2\cdots\alpha_{s-3/2}\dot\beta\dot\alpha_1\cdots\dot\alpha_{s-1/2}} = 0\ , \qquad s=\tfrac{3}{2},\tfrac{5}{2},... \tag{184}$$

The General Case:

Suppose that we have solved $R^1_{ab}(m+2,n-2)=0$ for the auxiliary gauge fields $\omega_{\alpha\dot\alpha}(m+1,n-1)$ in terms of $D_{[a}\omega_{b]}(m+2,n-2)$ and fixed the auxiliary gauge symmetries with parameters $\varepsilon(m,n)$ for $n\geq m+6\geq 6$ in the bosonic sector or $n\geq m+5\geq 5$ in the fermionic sector. We then turn to solving $R^1_{ab}(m+1,n-1)=0$. In doing so, we make use of the the Bianchi identity (161) for $R^1_{ab}(m+2,n-2)$ which reads

$$\xi(m+3,n-1) = \xi(m+1,n-1)+\eta(m+1,n-1) = \eta(m+1,n-3) = 0\ , \tag{185}$$

where ξ and η are the irreps that appear in the decomposition of $R^1_{ab}(m+1,n-1)$. Hence the remaining independent components of $R^1_{ab}(m+1,n-1)=0$ are given by

$$\xi(m+1,n-1)+\eta(m+1,n-1) = \xi(m-1,n-1) = \eta(m+1,n+1) = 0\ , \tag{186}$$

where the two linear combinations of $\xi(m+1,n-1)$ and $\eta(m+1,n-1)$ in (185) and (186) are independent. Since (186) reduces to (175), (180) and (183) we may as well define the range of m and n in (186) to be $|m-n|>1$, and it is straight forward to obtain:

$$\begin{aligned}\omega_{\alpha\dot\alpha,\beta_1\cdots\beta_m\dot\beta_1\cdots\dot\beta_n} &= \tfrac{1}{2(m+1)}D\omega_{\dot\beta_1\dot\beta_2,\alpha\beta_1\cdots\beta_m\dot\alpha\dot\beta_1\cdots\dot\beta_{n-2}}\\ &+\epsilon_{\dot\alpha\dot\beta_1}\tfrac{n}{2(n+1)}\Big[\tfrac{n-1}{m+n+2}D\omega_{\dot\beta_2}{}^{\dot\gamma}{}_{,\alpha\beta_1\cdots\beta_m\dot\gamma\dot\beta_3\cdots\dot\beta_n}\end{aligned}$$

$$
\begin{aligned}
&+\tfrac{n+1}{(m+1)(m+n+2)}\left(D\omega_{\alpha}{}^{\gamma}{}_{,\gamma\beta_1\cdots\beta_m\dot\beta_2\cdots\dot\beta_n}+m\,D\omega_{\beta_1}{}^{\gamma}{}_{,\gamma\alpha\beta_2\cdots\beta_m\dot\beta_2\cdots\dot\beta_n}\right)\\
&-\tfrac{m}{(m+1)(m+2)}\epsilon_{\alpha\beta_1}D\omega^{\gamma\delta}{}_{,\gamma\delta\beta_2\cdots\beta_m\dot\beta_2\cdots\dot\beta_n}\Big]\\
&+m\,\epsilon_{\alpha\beta_1}\eta_{\beta_2\cdots\beta_m\dot\alpha\dot\beta_1\cdots\dot\beta_n}\ ,\qquad |m-n|>1\ ,
\end{aligned}
\tag{187}
$$

where $\eta(m-1,n+1)$ is undetermined. Notice that $R^1_{ab}(1,n-1)=0$ determines $\omega_{\alpha\dot\alpha}(0,n)$ uniquely for $n\geq 2$. From (174) it follows that the transformation of $\eta(m-1,n+1)$ under the gauge symmetry with parameter $\varepsilon(m-1,n+1)$ is given by

$$
\delta\eta(m-1,n+1)\ =\ \varepsilon(m-1,n+1)\ ,\qquad |m-n|>1\ . \tag{188}
$$

This gauge symmetry can thus be fixed uniquely by imposing the algebraic gauge condition

$$
\eta(m-1,n+1)\ =\ 0\ ,\qquad |m-n|>1\ . \tag{189}
$$

Hence the gauge fields in (187) and the gauge symmetries used to impose (189) are auxiliary.

Summary of Dynamical Fields, Gauge Symmetries and Equations of Motion

We thus conclude that the dynamical degrees of freedom of the higher spin theory and the corresponding dynamical equations of motion and gauge symmetries are:

i) the generalized vierbeins $\omega(s-1,s-1,\theta)$ $(s=2,3,...)$ obeying the second order equations (181), with invariance under the generalized reparametrizations and generalized Lorentz transformations given by

$$
\begin{aligned}
\delta\omega_{\alpha\dot\alpha,\beta_1\cdots\beta_{s-1}\dot\beta_1\cdots\dot\beta_{s-1}} &= D_{\alpha\dot\alpha}\varepsilon_{\beta_1\cdots\beta_{s-1}\dot\beta_1\cdots\dot\beta_{s-1}}+(s-1)\epsilon_{\alpha\beta_1}\varepsilon_{\beta_2\cdots\beta_{s-1}\dot\alpha\dot\beta_1\cdots\dot\beta_{s-1}}\\
&\quad+(s-1)\epsilon_{\dot\alpha\dot\beta_1}\varepsilon_{\alpha\beta_1\cdots\beta_{s-1}\dot\beta_2\cdots\dot\beta_{s-1}}\ ,
\end{aligned}
\tag{190}
$$

ii) the generalized gravitini fields $\omega(s-\frac{3}{2},s-\frac{1}{2},\theta)(s=\frac{3}{2},\frac{5}{2},...)$ and their hermitian conjugates obeying the first order equations (184) with invariance under the generalized local supersymmetries and the local fermionic transformations given by

$$
\begin{aligned}
\delta\omega_{\alpha\dot\alpha,\beta_1\cdots\beta_{s-3/2}\dot\beta_1\cdots\dot\beta_{s-1/2}} &= D_{\alpha\dot\alpha}\varepsilon_{\beta_1\cdots\beta_{s-3/2}\dot\beta_1\cdots\dot\beta_{s-1/2}}+(s-\tfrac{3}{2})\epsilon_{\alpha\beta_1}\varepsilon_{\beta_2\cdots\beta_{s-3/2}\dot\alpha\dot\beta_1\cdots\dot\beta_{s-1/2}}\\
&\quad+(s-\tfrac{1}{2})\epsilon_{\dot\alpha\dot\beta_1}\varepsilon_{\alpha\beta_1\cdots\beta_{s-3/2}\dot\beta_2\cdots\dot\beta_{s-1/2}}\ ,
\end{aligned}
\tag{191}
$$

iii) the two $SO(8)$, spin $s=1$ gauge fields $\omega_{ij}(0,0)$ and $\omega_{i_1\cdots i_6}(0,0)$ transforming as

$$\delta\omega_{ij} \;=\; d\varepsilon_{ij} \;, \qquad \delta\omega_{i_1\cdots i_6} \;=\; d\varepsilon_{i_1\cdots i_6} \;, \tag{192}$$

the spin $s=\frac{1}{2}$ fermions C^{ijk}_{α}, $C^{i_1\cdots i_7}_{\alpha}$ and their hermitian conjugates and the scalars ϕ and ϕ_{ijkl} defined in (59) and obeying the reality condition (60). The fields with $s \leq 1$ obey equations of motion obtainable from (170) and (171).

6.2 The Analysis of the Spin $s \geq \frac{3}{2}$ Equations of Motion

The Bosonic Sector

In the bosonic case, the linearized equations of motion for the generalized vierbein $\omega_{\alpha\dot\alpha}(s-1,s-1)$ $(s\geq 2)$ is given by (181). Combining this equation with (154) we find

$$\begin{aligned} e_{0\,\alpha_1}{}^{\dot\beta} \wedge \Big[D\omega_{\alpha_2\cdots\alpha_{s-1}\dot\beta\dot\alpha_1\cdots\dot\alpha_{s-1}} + e_{\dot\beta}{}^{\beta} \wedge \omega_{\beta\alpha_2\cdots\alpha_{s-1}\dot\alpha_1\cdots\dot\alpha_{s-1}} & \\ +(s-1)e_{\dot\alpha_1}{}^{\beta} \wedge \omega_{\beta\alpha_2\cdots\alpha_{s-1}\dot\beta\dot\alpha_2\cdots\dot\alpha_{s-1}}\Big] \;&=\; 0 \;, \qquad s \;=\; 2,3,4,... \end{aligned} \tag{193}$$

Combining (193) with the expression (176) for the generalized Lorentz connection $\omega_{\alpha\dot\alpha}(s-2,2)$ in terms of the generalized vierbein and the gauge condition (177) we find that the linearized equations are second order in derivatives and that they are invariant under the gauge transformations (190). In order to determine the spectrum we shall use the gauge invariance to fix a gauge in which the equations reduce to a Klein-Gordon like equation $(D^2+M^2(s))\eta(s,s)=0$ where $\eta(s,s)$ is the spin s irrep of the generalized vierbein and $M^2(s)$ a critical AdS-mass such that the equation possesses a residual, on-shell gauge symmetry which leaves only two physical modes in $\eta(s,s)$. These two modes correspond to the "massless" representations of AdS group $SO(3,2)$. As for the ordinary photon representations of the Poincaré algebra these massless representations show the characteristic feature of multiplet shortening (where the on-shell gauge modes correspond to the null-states of the representation modules). The requirement of multiplet shortening amounts to an s dependent condition on the $SO(3,2)$ Casimir $C_2[SO(3,2)]$ which when turned into a differential operator acting on the modes of the higher spin fields generates the critical value of the mass term $M^2(s)$.

The gauge symmetries (190) allow us to fix the generalized Lorentz type gauge

$$D^{\alpha\dot\alpha}\omega_{\alpha\dot\alpha,\beta_1\cdots\beta_{s-1}\dot\beta_1\cdots\dot\beta_{s-1}} \;=\; 0 \;, \tag{194}$$

$$\omega_{\dot\alpha}{}^{\beta}{}_{,\beta\beta_1\cdots\beta_{s-2}\dot\beta_1\cdots\dot\beta_{s-1}} \;=\; 0 \;, \qquad s=2,3,... \tag{195}$$

The algebraic condition (195) eliminates all Lorentz irreps in the generalized vierbein except the (s,s) irrep. The gauge condition (194) and the $(s-2,s-2)$ component of (195) fix the generalized reparametrizations up to residual gauge transformations with parameters obeying

$$[D^2+1-s^2]\varepsilon_{\beta_1\cdots\beta_{s-1}\dot\beta_1\cdots\dot\beta_{s-1}} \;=\; 0\;, \qquad D^{\alpha\dot\alpha}\varepsilon_{\alpha\beta_1\cdots\beta_{s-2}\dot\alpha\dot\beta_1\cdots\dot\beta_{s-2}} \;=\; 0\;. \tag{196}$$

The residual gauge transformations also involve compensating generalized local Lorentz transformations with parameters given by

$$\varepsilon_{\alpha_1\cdots\alpha_{s-2}\dot\alpha_1\cdots\dot\alpha_s} \;=\; \tfrac{1}{s}D^{\beta}{}_{(\dot\alpha_1}\varepsilon_{|\beta\alpha_1\cdots\alpha_{s-2}|\dot\alpha_2\cdots\dot\alpha_s)}\;, \tag{197}$$

such that the first derivatives of the parameters of the generalized reparametrization are only subject to the second condition in (196). The $(s-2,s)$ component of (195) fixes uniquely the generalized Lorentz gauge parameter $\varepsilon(s-2,s)$. For $s=2$ and $\theta=0$ the gauge condition (194) yields transversality condition $D^{\mu}\omega_{\mu,b}$ on the graviton field $\omega_{\mu,a}$, the $(0,0)$ component of (195) yields the tracelessness condition $\eta^{ab}\omega_{a,b}$ and the $(0,2)$ component of (195) implies the Lorentz gauge $\omega_{[a,b]}=0$.

From (194) and (195) it follows that the derivatives of the generalized vierbein obey

$$D\omega_{\alpha}{}^{\beta}{}_{,\beta\beta_1\cdots\beta_{s-2}\dot\beta_1\cdots\dot\beta_{s-1}} \;=\; 0\;, \tag{198}$$

which implies that $D\omega_{\alpha\beta,\gamma_1\cdots\gamma_{s-1}\dot\gamma_1\cdots\dot\gamma_{s-1}}$ is symmetric in all its undotted indices. The solution (176) for the generalized Lorentz connection then simplifies to

$$\omega_{\alpha_1\dot\alpha_1,\alpha_2\cdots\alpha_{s-1}\dot\alpha_2\cdots\dot\alpha_{s+1}} \;=\; \tfrac{2}{s-1}D\omega_{\dot\alpha_1\dot\alpha_2,\alpha_1\cdots\alpha_{s-1}\dot\alpha_3\cdots\dot\alpha_{s+1}}\;. \tag{199}$$

After a straightforward calculation, where one repeatedly makes use of (194-195), (198) and the expression (126) for the Lorentz curvature, one finds

$$\left[D^2+3-(s-1)^2\right]\omega_{\alpha_1\dot\alpha_1,\alpha_2\cdots\alpha_s\dot\alpha_2\cdots\dot\alpha_s} \;=\; 0\;. \tag{200}$$

This equation describes an irreducible, massless spin s field $\omega_{\alpha_1\dot\alpha_1,\alpha_2\cdots\alpha_s\dot\alpha_2\cdots\dot\alpha_s}$ with AdS-energy

$$E_0 \;=\; s+1\;. \tag{201}$$

To verify (201) we consider the harmonic expansion on the coset $SO(3,2)/SO(3,1)$. The positive energy representations of $SO(3,2)$ can be characterized in the $SO(3)\times SO(2)$ basis, where $SO(3)$ is generated by the spatial rotations M_{ij} $(i,j=1,2,3)$ and $SO(2)$ by the energy operator M_{04}. These representations are labeled by the lowest energy E_0 and the highest eigenvalue s_0 of M_{12} when the energy is fixed to be E_0. Following the procedure described in [1, 5], we Euclideanize the AdS group to $SO(5)$, with irreps labeled by highest weights (n_1,n_2), and the Lorentz group to $SO(4)$, with irreps labeled by highest weights (j_1,j_2). The quadratic Casimir eigenvalues for these groups are

$$C_2[SO(3,2)] = E_0(E_0-3)+s(s+1) \ , \tag{202}$$

$$C_2[SO(5)] = n_1(n_1+3)+n_2(n_2+1) \ . \tag{203}$$

This suggests that in continuing $SO(5)$ back to $SO(3,2)$ we identify n_1 with $-E_0$ and n_2 with s_0. Since the $SO(4)$ content of the gauge fixed generalized vierbein $\omega_{\alpha_1\dot\alpha_1,\alpha_2\cdots\alpha_s\dot\alpha_2\cdots\dot\alpha_s}$ is $(j_1,j_2)=(s,0)$ we can expand $\omega_{\alpha_1\dot\alpha_1,\alpha_2\cdots\alpha_s\dot\alpha_2\cdots\dot\alpha_s}$ in terms of representation functions of $SO(5)$ as follows

$$\omega_{\alpha_1\dot\alpha_1,\alpha_2\cdots\alpha_s\dot\alpha_2\cdots\dot\alpha_s}(x) = \sum_{n_1\geq s\geq n_2\geq 0}\sum_p \omega_p^{(n_1n_2)} D^{(n_1n_2)}_{\alpha_1\cdots\alpha_s\dot\alpha_1\cdots\dot\alpha_s,\,p}(L_x^{-1}) \ , \tag{204}$$

where $\omega_p^{(n_1n_2)}$ are constant expansion coefficients, $D^{(n_1n_2)}_{\alpha_1\cdots\alpha_s\dot\alpha_1\cdots\dot\alpha_s,\,p}(L_x^{-1})$, known as Wigner functions, refer to the representation of the coset representative L_x^{-1} with rows labeled by $\alpha_1\cdots\alpha_s$ and $\dot\alpha_1\cdots\dot\alpha_s$ and columns by $p=1,...,dim(n_1n_2)$. The eigenvalues of the d'Alembertian acting on the Wigner functions are computed from the formula

$$D^2 L_x^{-1} = -\lambda^2\Big(C_2[SO(5)]-C_2[SO(4)]\Big)L_x^{-1} \ , \tag{205}$$

where D^2 is the d'Alembertian in the Euclidean metric (which have opposite sign to the AdS d'Alembertian) and the quadratic Casimir of $SO(4)$ is given by

$$C_2[SO(4)] = j_1(j_1+2)+j_2^2 \ . \tag{206}$$

Thus, using that in continuing back to AdS one has to let $D^2\to -D^2$ and $n_1\to -E_0$, and setting the inverse AdS radius $\lambda=1$, we find from (200) that the energy eigenvalues are to be solved from the characteristic equation

$$E_0(E_0-3)-s+3-(s-1)^2 = 0 \ , \tag{207}$$

with the positive energy solution (201).

To calculate the number of massless modes we notice that the gauge transformations generated by the residual parameters obeying (196) obey the gauge fixed equations of motion (200). Hence the number of real on-shell degrees of freedom is given by the number of components of the spin s irrep ($(s+1)^2$), minus the number of gauge conditions (194) linear in derivatives (s^2), minus the number of residual gauge symmetries, which is equal to the number of degrees of freedom in $\varepsilon(s-1,s-1)$ (s^2) minus the number of constraints (196) linear in derivatives ($(s-1)^2$). Thus there are

$$(s+1)^2-s^2-[s^2-(s-1)^2] = 2 \ , \tag{208}$$

on-shell degrees of freedom with spin $s = 2, 3, 4, ...$ and energy $E_0 = s + 1$ describing massless higher spin bosons.

The Fermionic Sector

In the fermionic case, we combine the equation (184) with (154) to obtain the following first order equation for the generalized gravitini fields $\omega(s-\frac{3}{2}, s-\frac{1}{2})$ and their hermitian conjugates $\omega(s-\frac{1}{2}, s-\frac{3}{2})$:

$$
\begin{aligned}
& e_{0\,\alpha_1}{}^{\dot\beta} \wedge \Big[D\omega_{\alpha_2\cdots\alpha_{s-1/2}\dot\alpha_1\cdots\dot\alpha_{s-1/2}} + e_{\dot\beta}{}^{\beta} \wedge \omega_{\beta\alpha_2\cdots\alpha_{s-1/2}\dot\alpha_1\cdots\dot\alpha_{s-3/2}} \\
& \qquad + (s-\tfrac{3}{2}) e_{\dot\alpha_1}{}^{\beta} \wedge \omega_{\beta\alpha_2\cdots\alpha_{s-1/2}\dot\beta\dot\alpha_2\cdots\dot\alpha_{s-3/2}} \Big] \;=\; 0 \;, \qquad s \;=\; \tfrac{3}{2}, \tfrac{5}{2}, \tfrac{7}{2}, ... \qquad (209)
\end{aligned}
$$

The local symmetries of this equation are given in (191) and they allow us to fix the following gauge

$$
\begin{aligned}
D^{\alpha\dot\alpha}\omega_{\alpha\dot\alpha,\beta_1\cdots\beta_{s-3/2}\dot\beta_1\cdots\dot\beta_{s-1/2}} &= 0 \;, & (210)\\
\omega_{\alpha}{}^{\dot\alpha}{}_{,\beta_1\cdots\beta_{s-3/2}\dot\alpha\dot\beta_1\cdots\dot\beta_{s-3/2}} &= 0 \;, & (211)\\
\omega^{\alpha}{}_{\dot\beta_1,\alpha\beta_1\cdots\beta_{s-5/2}\dot\beta_2\cdots\dot\beta_{s+1/2}} &= 0 \;. & (212)
\end{aligned}
$$

Eqs. (210) and (211) fix the generalized local supersymmetries up to residual symmetries generated by parameters obeying the Dirac equation

$$
D_\alpha{}^{\dot\alpha}\varepsilon_{\beta_1\cdots\beta_{s-3/2}\dot\alpha\dot\beta_1\cdots\dot\beta_{s-3/2}} - (s+\tfrac{1}{2})\,\varepsilon_{\alpha\beta_1\cdots\beta_{s-3/2}\dot\beta_1\cdots\dot\beta_{s-3/2}} \;=\; 0 \;. \qquad (213)
$$

The residual supersymmetry transformations also involve a compensating fermionic gauge transformation with parameter $\varepsilon(s-\frac{5}{2}, s+\frac{1}{2})$ given by

$$
\varepsilon_{\alpha_1\cdots\alpha_{s-5/2}\dot\alpha_1\cdots\dot\alpha_{s+1/2}} \;=\; \frac{1}{s-\frac{1}{2}} D^{\beta}{}_{(\dot\alpha_1}\varepsilon_{|\beta\alpha_1\cdots\alpha_{s-5/2}|\dot\alpha_2\cdots\dot\alpha_{s+1/2})} \;, \qquad (214)
$$

The gauge condition (212) fixes uniquely the gauge parameters $\varepsilon(s-\frac{5}{2}, s+\frac{1}{2})$. Together (211) and (212) eliminate all Lorentz irreps in the generalized gravitino except the spin s irrep.

The gauge choice allows us to rewrite the fermionic equation (209) as the generalized Dirac equation

$$
D_\alpha{}^{\dot\alpha}\omega_{\beta_1\dot\alpha,\beta_2\cdots\beta_{s-1/2}\dot\beta_1\cdots\dot\beta_{s-1/2}} - (s-\tfrac{1}{2})\omega_{\alpha\dot\beta_1,\beta_1\cdots\beta_{s-1/2}\dot\beta_2\cdots\dot\beta_{s-1/2}} \;=\; 0 \;. \qquad (215)
$$

By combining (215) with its hermitian conjugate and making repeated use of (210-212) we obtain the second order equation

$$\left[D^2 + s + \tfrac{5}{2} - (s - \tfrac{1}{2})^2\right] \omega_{\alpha_1\dot{\alpha}_1,\alpha_2\cdots\alpha_{s-1/2}\dot{\alpha}_2\cdots\dot{\alpha}_{s+1/2}} \;=\; 0\ . \tag{216}$$

Applying the techniques for spectral analysis described in the bosonic xase, we find that the Euclideanized, gauge fixed generalized gravitino belongs to the $(j_1 j_2) = (s, -\frac{1}{2})$ representation of $SO(4)$ (and its hermitian conjugate belongs to $(j_1 j_2) = (s, \frac{1}{2})$). The harmonic expansion now involves $SO(5)$ irreps satisfying $n_1 \geq s \geq n_2 \geq \frac{1}{2}$. The AdS energies therefore solve the characteristic equation

$$E_0(E_0 - 3) + \tfrac{9}{4} - (s - \tfrac{1}{2})^2 \;=\; 0\ , \tag{217}$$

which has the positive energy root $E_0 = s + 1$.

To find the number of massless modes we verify that the gauge transformations generated by the residual parameters obeying (213) also obey the Dirac equation (215). Hence the count of on-shell degrees of freedom (following rules analogous to those given for the bosonic count and taking into account the fact that the Dirac operator in (215) has half the maximum rank) shows that there are

$$(s + \tfrac{1}{2})(s + \tfrac{3}{2}) - (s - \tfrac{1}{2})(s + \tfrac{1}{2}) - \left[(s - \tfrac{1}{2})(s + \tfrac{1}{2}) - (s - \tfrac{3}{2})(s - \tfrac{1}{2})\right] \;=\; 2 \tag{218}$$

real, on-shell degrees of freedom with spin $s = \frac{3}{2}, \frac{5}{2}, \frac{7}{4}, \ldots$ and energy $E_0 = s + 1$ describing massless higher spin fermions.

6.3 The Spin $s \leq 1$ Sector

In this sector the equations of motion follows from (170). In the spin $s = 1$ sector we obtain

$$D_a R^1_{bc}(\theta) \;=\; \tfrac{i}{4}(\sigma_{bc})^{\alpha\beta}(\sigma_a)^{\gamma\dot{\delta}} C_{\alpha\beta\gamma\dot{\delta}}(\theta) + \tfrac{i}{4}(\bar{\sigma}_{bc})^{\dot{\alpha}\dot{\beta}}(\bar{\sigma}_a)^{\dot{\gamma}\delta} C_{\delta\dot{\alpha}\dot{\beta}\dot{\gamma}}(\theta)\ . \tag{219}$$

from the $(m, n) = (2, 0)$ component of (159), the identity $C_{\alpha\beta}(\theta) = R^1_{\alpha\beta}(\theta)$ and (160). Multiplying this equation with η^{ab} and ϵ^{abcd}, and using the "membrane" identities

$$(\sigma^{ab})^{(\alpha\beta}(\sigma_a)^{\gamma)\dot{\delta}} \;=\; 0\ , \qquad \epsilon^{abcd}(\sigma_{bc})^{(\alpha\beta}(\gamma_d)^{\gamma)\dot{\delta}} \;=\; 0\ , \tag{220}$$

and expanding the θ dependence using (35), one finds the spin 1 equations of motion and the Bianchi identities:

$$D^a R^{1\,ij}_{ab} \;=\; 0\ , \qquad \epsilon^{abcd} D_b R^{1\,ij}_{cd} \;=\; 0\ ,$$

$$D^a R^{1\, i_1 \cdots i_6}_{ab} \;=\; 0\;, \qquad \epsilon^{abcd} D_b R^{1\, i_1 \cdots i_6}_{cd} \;=\; 0\;. \tag{221}$$

The gauge transformations $\delta\omega(\theta) = d\epsilon(\theta)$ allow us to impose the Lorentz gauge $D^a \omega_a(\theta) = 0$ in which the equations of motion take the form

$$\Big(D^2 + 3\Big)\, \omega^{ij}_a \;=\; 0\;, \qquad \Big(D^2 + 3\Big)\, \omega^{i_1 \cdots i_6}_a \;=\; 0\;. \tag{222}$$

The corresponding characteristic equation has the critical root $E_0 = 2$ and the residual gauge symmetries as usual cancel the longitudinal on-shell mode. Hence the theory contains two massless $SO(8)$ vector fields (gauging the $SO(8)_+$ and $SO(8)_-$ discussed in section 2).

In the spin $s = \frac{1}{2}$ sector the equations of motion are given by the $(m, n) = (1, 0)$ component of (159):

$$D_{\alpha\dot\alpha} C_\beta(\theta) \;=\; i\, C_{\alpha\beta\dot\alpha}(\theta)\;. \tag{223}$$

Expanding the θ dependence of the $(0, 1)$ component of this equation one finds the first order Dirac equations

$$D^\alpha{}_{\dot\alpha} C^{ijk}_\alpha \;=\; 0\;, \qquad D^\alpha{}_{\dot\alpha} C^{i_1 \cdots i_7}_\alpha \;=\; 0\;, \tag{224}$$

giving two real, on-shell fermionic degrees freedom. Squaring these equations gives

$$\Big(D^2 + 3\Big)\, C^{ijk}_\alpha \;=\; 0\;, \qquad \Big(D^2 + 3\Big)\, C^{i_1 \cdots i_7}_\alpha \;=\; 0\;, \tag{225}$$

with critical energy $E_0 = \frac{3}{2}$, as expected for massless spin $\frac{1}{2}$ fermion fields.

Finally in the scalar sector the $(m, n) = (0, 0)$ and $(1, 1)$ components of (159) read

$$D_{\alpha\dot\alpha} \phi(\theta) \;=\; i\, \phi_{\alpha\dot\alpha}(\theta)\;, \qquad D_{\beta\dot\beta} \phi_{\alpha\dot\alpha}(\theta) \;=\; -i\, \epsilon_{\beta\alpha} \epsilon_{\dot\beta\dot\alpha} \phi(\theta) + i\, \phi_{\alpha\beta\dot\alpha\dot\beta}(\theta)\;, \tag{226}$$

where $\phi(\theta)$ comprise the complex scalar fields introduced in (59). Evaluating the $D^{\alpha\dot\alpha}$ divergence of the first equation in (226) using the latter equation and expanding in θ yields the scalar field equations

$$\Big(D^2 + 2\Big)\, \phi \;=\; 0\;, \qquad \Big(D^2 + 2\Big)\, \phi^{ijkl} \;=\; 0\;, \tag{227}$$

with critical energies $E_0 = 1$ and $E_0 = 2$, as expected for massless scalars.

7 The Linearized $N = 8$ AdS Supergravity

We expect the linearized field equations and supersymmetry transformations of the level $k = 0$ multiplet of Table 1 to agree with those of gauged $N = 8$ supergravity. Let us derive the exact correspondence and relate the coupling constants of the higher spin theory to the gravitational coupling constant κ and the $SO(8)$ gauge coupling g of the $N = 8$ theory.

The level $k = 0$, spin $s \leq \frac{1}{2}$ equations are given by (221), (224) and (227), while the gauge invariant spin $s = \frac{3}{2}$ equation is given by (209). In the spin $s = 2$ sector the linearized equation of motion follow from the spin $s = 2$ component of (167) and the constraints listed in (168), i.e. the $9 + 1$ real components

$$R^1_{\alpha\beta,\dot\alpha\dot\beta} \;=\; R^1_{\dot\alpha\dot\beta,\alpha\beta} \;=\; 0 \;, \qquad R^1_{\alpha\beta,\alpha\beta} \;=\; R^1_{\dot\alpha\dot\beta,\dot\alpha\dot\beta} \;=\; 0 \;. \tag{228}$$

From the discussion following (164) we recall that the torsion constraint given by the spin $s = 2$ component of (167) together with the Bianchi identity (161) imply the reality of the quantities in (228) as well as the vanishing of the 6 real components in

$$R^1_{\alpha_1}{}^{\beta}{}_{,\alpha_2\beta} \;=\; R^1_{\dot\alpha_1}{}^{\dot\beta}{}_{,\dot\alpha_2\dot\beta} \;=\; 0 \;. \tag{229}$$

In order to compare the spin $s = 2$ equation to the $N = 8$ theory it is convenient to rewrite it as the linearization of Einstein's vacuum equation with a cosmological constant (including higher orders to the spin $s = 2$ field equation will of course lead to more complicated terms in the right side of the Einstein equation). To this end we define the Ricci tensor r_{ab} in the usual way as

$$r_{ab} \;=\; e_a{}^{\mu} e_c{}^{\nu} r_{\mu\nu,b}{}^{c} \;, \tag{230}$$

where $r_{\mu\nu\,ab}$ is the $SO(3,1)$-valued Riemann tensor and $e_{a\mu}$ denotes the inverse of the vierbein $e_{\mu a}$ defined in (257). Linearizing (230) around the AdS vacuum (126), we find

$$r_{ab} \;=\; \Lambda\eta_{ab} + r^1_{ab} \;, \qquad \Lambda \;=\; -3\lambda^2 \;, \tag{231}$$

$$r^1_{ab} \;=\; r^1_{ac,b}{}^{c} + 2\,\omega_{a,b} + \eta_{ab}\,\omega^{c}{}_{,c} \;, \tag{232}$$

where λ is the inverse AdS radius defined in (124), and the linearized Riemann curvature

$$r^1_{\mu\nu,ab} \;=\; 2D_{[\mu}\omega_{\nu],ab} \;, \tag{233}$$

where D_μ now denotes the background Lorentz covariant derivative. To obtain the Einstein equation from the constraints it is more convenient to treat the Lorentz connection as an independent field (rather than substituting the solution for the Lorentz connection obtained from the torsion constraint into the Riemann tensor). The Ricci tensor (230) then contains 16 real components constrained by the now independent components in (228) and (229). From (160) it follows that the linearized, AdS covariant curvature $R^1_{ab,cd}$ and the linearized Riemann tensor $r^1_{ab,cd}$ are related by

$$
\begin{aligned}
R^1_{\alpha_1\alpha_2,\beta_1\beta_2} &= r^1_{\alpha_1\alpha_2,\beta_1\beta_2} - 2\,\epsilon_{\alpha_1\beta_1}\omega_{\alpha_2}{}^{\dot\gamma}{}_{\beta_2\dot\gamma}\ , \\
R^1_{\alpha_1\alpha_2,\dot\beta_1\dot\beta_2} &= r^1_{\alpha_1\alpha_2,\dot\beta_1\dot\beta_2} - 2\,\omega_{\alpha_1\dot\beta_1,\alpha_2\dot\beta_2}\ ,
\end{aligned}
\tag{234}
$$

and hermitian conjugates, where

$$
\begin{aligned}
r^1_{\alpha\beta,\gamma\delta} &= \tfrac{1}{8}(\sigma^{ab})_{\alpha\beta}(\sigma^{cd})_{\gamma\delta}r_{ab,cd}\ , \\
r^1_{\alpha\beta,\dot\gamma\dot\delta} &= \tfrac{1}{8}(\sigma^{ab})_{\alpha\beta}(\bar\sigma^{cd})_{\dot\gamma\dot\delta}r_{ab,cd}\ , \\
\omega_{\alpha\dot\alpha,\beta\dot\beta} &= -\tfrac{1}{2}(\sigma^a)_{\alpha\dot\alpha}(\sigma^b)_{\beta\dot\beta}\omega_{a,b}\ ,
\end{aligned}
\tag{235}
$$

as follows from (162) and (257). The constraints (228) and (229) then yield

$$
\begin{aligned}
r^1_{c\{a,b\}}{}^c - 2\omega_{\{a,b\}} &= 0\ , \\
r^1_{ab,}{}^{ab} + 6\omega^a{}_{,a} &= 0\ , \\
r^1_{c[a,b]}{}^c - 2\omega_{[a,b]} &= 0\ ,
\end{aligned}
\tag{236}
$$

where $\{ab\}$ denotes the traceless symmetric part. In deriving (236) one has to make use of the self-duality properties (247) and notice that the pairs of indices ab and cd of the Riemann tensor in the right side of in the two first equations in (235) are projected onto (anti-)selfdual components. Eqs. (236) then follow by adding up selfdual and anti-selfdual components of the constraints (228) and (229). Combining (232) and (236) yields $r^1_{ab} = 0$, that is, in the linearized approximation the equations of motion for the spin $s = 2$ vierbein can be written as the Einstein's equation with cosmological constant.

In summary, the linearized equations of motion of the level $k = 0$ multiplet are given by

$$
\begin{aligned}
&\text{spin } s=2 \ : && r_{ab} = -3\lambda^2\,\eta_{ab}\ , \\
&\text{spin } s=\tfrac{3}{2} \ : && (\sigma^{abc})_\alpha{}^{\dot\alpha}\left(D_b\,\omega^i_{c,\dot\alpha} - \tfrac{1}{2}\lambda\,(\sigma_b)_{\dot\alpha}{}^\beta\omega^i_{c,\beta}\right) = 0\ , \\
&\text{spin } s=1 \ : && D^aD_{[a}\omega^{ij}_{b]} = 0\ , \\
&\text{spin } s=\tfrac{1}{2} \ : && D_\alpha{}^{\dot\alpha}C^{ijk}_{\dot\alpha} = 0\ , \\
&\text{spin } s=0 \ : && \left(D^2 + 2\lambda^2\right)\phi^{ijkl} = 0\ ,
\end{aligned}
\tag{237}
$$

where we have reintroduced the inverse AdS radius λ also in the spin $s \leq \frac{3}{2}$ equation.

The linearized, gauged $N=8$ supergravity model is described by the quadratic action [32]

$$
\begin{aligned}
e^{-1}\mathcal{L}_2 \;\; = \;\; & \frac{1}{2}R - \frac{\kappa^2}{8g^2}F_{\mu\nu,IJ}F^{\mu\nu,IJ} - \frac{1}{96}\partial_\mu\phi_{ijkl}\partial^\mu\phi^{ijkl} + \frac{g^2}{24\kappa^2}\,\phi_{ijkl}\phi^{ijkl} + \frac{6g^2}{\kappa^2} \\
& + \Big(\tfrac{i}{2}\bar{\psi}^i_{L\mu}\gamma^{\mu\nu\rho}D_\nu\psi_{L\rho i} + \frac{ig}{\sqrt{2}\kappa}\bar{\psi}^i_{R\mu}\gamma^{\mu\nu}\psi_{L\nu,i} - \frac{1}{2}\bar{\chi}^{ijk}_L\gamma^\mu D_\mu\chi_{Lijk} \;+ h.c.\Big) \;, \quad (238)
\end{aligned}
$$

where $i,j,..=1,...,8$ are $SU(8)$ indices and $I,J,..=1,...,8$ are $SO(8)$ indices, ϕ_{ijkl} are the $35+35$ scalars obeying (60) and the fermions are Weyl. The complex conjugation changes chirality, and consequently both chiralities occur for the gravitini as well as the spin $1/2$ fields. Thus, the theory is vector like. In writing (238) we have assigned (energy) dimension $\frac{1}{2}$ to all the fermions, dimension 0 to the scalars and dimension 1 to the vector fields. Thus the Lagrangian $\mathcal{L}_2$ has dimension 2.

We find that (237) is in perfect agreement with (238) provided that we make the identifications

$$
\begin{aligned}
\omega_\mu{}^{\alpha\dot\alpha} \;\; &\rightarrow \;\; e_\mu{}^a \;, \qquad \omega^{ij}_\mu \;\rightarrow\; A^{IJ}_\mu \;, \\
\omega^i_{\mu\alpha} \;\; &\rightarrow \;\; \psi^i_{L\mu} \;, \qquad C^{ijk}_\alpha \;\rightarrow\; \chi^{ijk}_L \;, \qquad (239)
\end{aligned}
$$

and identify the following important relation between the Newton's constant κ, the $SO(8)$ gauge coupling g and the inverse AdS radius:

$$
\frac{g^2}{\kappa^2} \;\; = \;\; \tfrac{1}{2}\lambda^2 \;. \qquad (240)
$$

The free parameters of the higher spin theory are therefore the gauge coupling g and the inverse AdS radius λ. The gauge coupling is introduced into the full set of higher spin equations (96-100) and (101) by replacing $W \rightarrow g\,W$. These equations are consistent with the assignment of dimension 0 to the master fields W and Φ. Using the dimensionful coupling λ one then defines component fields $\tilde\omega_\mu(m,n;\theta)$ and $\tilde C(m,n;\theta)$ with canonical mass dimensions as follows [20]

$$
\begin{aligned}
\tilde\omega_\mu(m,n;\theta\,) \;\; &= \;\; \begin{cases} \lambda^{-1+\frac{|m-n|}{2}}\omega_\mu(m,n;\theta\,)\;, & (m,n)\neq(0,0)\;, \\ \omega_\mu(0,0,\theta\,)\;, & (m,n)=(0,0) \end{cases} \\
\tilde C(m,n;\theta\,) \;\; &= \;\; \lambda^{\frac{m+n}{2}}C(m,n;\theta\,)\;, \qquad m,n=0,1,... \qquad (241)
\end{aligned}
$$

Since the mass dimension of $\omega_\mu(m,n,\theta)$ is equal to 1 this means that the generalized vierbein $\tilde\omega_a(m,m;\theta\,)$ has mass dimension 0, the generalized gravitino $\tilde\omega_a(m-1,m;\theta\,)$ has dimension $\frac{1}{2}$

and the generalized Lorentz connection $\tilde{\omega}_a(m-2,m;\theta)$ and the $SO(8)$ vector fields $\tilde{\omega}_\mu(0,0,\theta)$ has dimension 1. The generalized Weyl tensor $\tilde{C}(m+2,0;\theta)$ $(m\geq 1)$ has dimension $\frac{m+2}{2}$, which is the same as the dimension of the pure curvature component $\tilde{R}_{\alpha\beta}(m,0;\theta)$. Finally the fermions $\tilde{C}^{ijk}_\alpha$ and $\tilde{C}^{i_1\cdots i_7}_\alpha$ hve dimension $\frac{1}{2}$ and the scalars $\tilde{\phi}$ and $\tilde{\phi}_{ijkl}$ have dimension 0. Thus, in particular, (241) yields the correct canonical dimensions of fields of the $N=8$ supergravity multiplet.

8 Discussion

The results of this paper suggest that the $D=4, N=8$ AdS supergravity can be embedded into a higher spin gauge theory. The fully nonlinear equations are consistent but we have shown the embedding at the linearized level. The important next step in this program is to study the interactions, starting with the quadratic fields in the equations of motion. Various aspects of such interactions have been studied before but they have not been compared to those of $N=8$ AdS supergravity. Given the facts that:

(a) the full higher spin equations of motion are consistent,

(b) they yield the correct $N=8$ AdS supergravity equations at the linearized level,

(c) the interactions in the $N=8$ AdS supergravity seem to be unique due to the highest possible supersymmetry in the theory,

one may expect that the higher spin equations at hand already contain the full fledged $N=8$ AdS supergravity, along with the sector for the spin $s\geq\frac{5}{2}$ fields. If so, then one would also expect to uncover the $E_7/SU(8)$ coset structure of Cremmer-Julia [34, 35] which plays an important role in the description of both the $N=8$ Poincaré [34, 35] as well as the AdS $N=8$ supergravity [32, 33].

Relevant to the problem of finding the hidden symmetries in the theory is the question of how unique is the higher spin AdS supergravity. This issue has already been addressed in section 4.3. The full significance of an interaction ambiguity discussed in that section is not clear to us at present. It may as well play a role in the search for the hidden E_7 symmetry. Recalling the uniqueness of the $N=8$ AdS supergravity, we expect that there should be no ambiguity in the interactions provided that we insist on the consistent truncation of the theory to the pure $N=8$ AdS supergravity. A careful comparison of the first interaction terms in the higher spin theory and the $N=8$ AdS supergravity is required to settle this question.

The $N=8$ supersingleton propagating at the boundary of AdS spacetime serves as a spectrum generating representation for the massless higher spin theory propagating in the bulk of the AdS spacetime. It is tempting to believe that the singletons could play a more fundamental role in the derivation of the effective bulk action from the bulk/boundary duality prescription of [31, 38, 39]. Notice that both bulk theory and boundary theory has the same dimensionful coupling λ(the inverse AdS radius). Since the (massless) spectra of the bulk and the boundary theories agree, the essential test of the bulk/boundary duality is therefore whether it is possible to represent the $shs^E(8|4)$ symmetry algebra as charges of conserved currents in the $N = 8$ supersingleton theory. In that case we would expect that the nonlinearities of the bulk theory would be reproduced by the interactions between the composite singleton states (the dimensionless gauge coupling g would the be introduced in the boundary theory by a rescaling of the composite states describing the gauge fields). While this program by and large remains to be realized, the construction of higher spin currents has been recently investigated [45], at least for low lying spins, in the context of AdS_5. Higher spin supercurrents have also been constructed [46], also in the context of AdS_5.

Since the boundary theory involves massless as well as massive composite states, bulk/boundary duality would also yield higher spin bulk interactions including both massless and massive "matter" sectors (allowing a consistent truncation to the massless sector). Inclusion of massive sectors could generate mechanisms for spontaneous breaking [27, 11] of the $shs^E(8|4)$ symmetry in which the massive multiplets are "eaten" by the massless gauge multiplets. This point is of great physical interest since we do not presently know how to fit massless higher spins into an M-theoretical framework.

A desirable formal consequence of bulk/boundary duality would be to ultimately reduce the rather cumbersome calculations implied by the perturbative expansion around the AdS vacuum outlined in section 5.2 to calculations entirely within the free boundary supersingleton field theory. An interesting related issue is whether it is possible to accommodate the auxiliary spinor variables in the boundary theory and derive the higher spin (x, Z) space field equations (96-100) using background field methods.

The bulk/boundary duality of the type discussed above may also exist for the $4D$ doubletons (vector multiplet) propagating at the boundary of AdS_5 and the $6D$ doubletons (tensor multiplet) propagating at the boundary of AdS_7, respectively [40]. The possibility of constructing higher spin interactions in AdS spaces of dimension larger that four has been investigated in [41]. A more tractable example is the $3D$ higher spin theory described in [29]. In this case, the study of bulk/boundary duality is expected to be simpler because the $2D$ boundary theory is a more tractable conformal field theory. Another reason for the tractability of the $3D$ case is that it may be possible to construct an action for a theory based on the $3D$ higher spin AdS superalgebra, combining the elements of the work of [42] that involve a Chern-Simons action and the work described in [29].

It would, of course, be desirable to find an action which yields the consistent, fully nonlinear equations of motion of Vasiliev that we have studied here. Indeed, $R \wedge R$ type actions have been considered before [19, 20, 41, 23], but one drawback of these actions is that the spin $s \leq 1/2$ sector of the theory does not fit in a natural and geometrical way into the part of the action that describes the fields with spin $s \geq 3/2$. On the other hand, the fact that there is a gauge master field W and a matter master field Φ in the Vasiliev formalism studied here is suggestive of a $D = 5$ origin in which the matter master field may emerge as the fifth component of the gauge master field upon dimensional reduction to $D = 4$. Furthermore, it is encouraging that there exists the possibility of a Chern-Simons type Lagrangian of the form $tr(R \wedge R \wedge W)$ in five dimensions (the definition of traces of higher spin algebras is explained in [29]).

The construction of higher spin superalgebras and free field equations of motion for higher spin fields becomes rapidly very complicated in higher than four dimensions [41] and understandably not much progress has been made in this front for sometime. However, the recent exciting developments in M-theory provide ample motivation for exploring the $D = 11$ origin of the AdS higher spin supergravity studied here. Starting from the $D = 11$ supergravity theory alone, Kaluza-Klein compactification gives rise to massless and massive fields of maximum spin two. The corners of M-theory where it can be treated perturbatively, on the other hand, can give rise to infinite towers of higher spin fields, but all of these fields are massive for spin $s > 2$. Therefore, one is led to speculate about the existence of either a new corner of M-theory, or supermembrane theory, which may give rise to the higher spin AdS supergravity in $D = 4$ in a certain limit, or a new kind of $D = 11$ limit which modifies the well known $D = 11$ supergravity theory in a profound way. The first scenario is in line with the previous studies on the supermembrane-supersingleton connection [8, 9, 10, 11, 12, 13]. The latter scenario is motivated by a recent construction of the singleton/doubleton representations of a candidate $D = 11$ AdS supergroup in [43], which turn out to be of rather unusual kind. It is not known yet if an action, or equations of motion, can be written down to describe these representations in the ten dimensional boundary of AdS_{11}, but the possibility is certainly tantalizing and we expect that it would be highly relevant to the massless higher spin theory in the M-theory framework. In this context, it is interesting to note that the representations of $SO(9)$ group as the little group classifying the massless degrees of freedom of an eleven dimensional supergravity has been studied recently [44] with interesting results that hint at the possibility of higher spin massless fields.

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A Spinor Conventions

For $SO(3,1)$ we take $\eta_{ab} = \mathrm{diag}(-+++)$ and work with two-component Weyl spinors

$$\begin{aligned} y^\alpha &= \epsilon^{\alpha\beta} y_\beta \, , & y_\alpha &= y^\beta \epsilon_{\beta\alpha} \, , \\ \bar{y}_{\dot\alpha} &= (y_\alpha)^\dagger \, , & \bar{y}^{\dot\alpha} &= (y^\alpha)^\dagger \, , \\ \bar{y}^{\dot\alpha} &= \epsilon_{\dot\alpha\dot\beta} \bar{y}_{\dot\alpha} \, , & \bar{y}_{\dot\alpha} &= \bar{y}^{\dot\beta} \epsilon^{\dot\beta\dot\alpha} \, , \end{aligned} \tag{242}$$

where the charge conjugation matrix $\epsilon_{\alpha\beta} = \epsilon^{\alpha\beta} = \epsilon_{\dot\alpha\dot\beta} = \epsilon^{\dot\alpha\dot\beta}$ obeys $\epsilon_{\alpha\beta}\epsilon^{\alpha\gamma} = \delta^\gamma_\beta$. Using the Pauli matrices $\sigma^{1,2,3}$ we define the van der Waerden symbols $(\sigma^a)_{\alpha\dot\beta}$ $(a = 0,1,2,3)$

$$\begin{aligned} (\sigma^a)_{\alpha\dot\beta} &:= (1, \sigma^1, \sigma^2, \sigma^3) \, , \\ (\bar\sigma^a)^{\dot\alpha\beta} &:= (1, -\sigma^1, -\sigma^2, -\sigma^3) = \epsilon^{\dot\alpha\dot\gamma}\epsilon^{\beta\delta}(\sigma^a)_{\delta\dot\gamma} \, , \end{aligned} \tag{243}$$

with the hermicity properties

$$\Big((\sigma^a)_{\alpha\dot\beta}\Big)^\dagger = (\bar\sigma^a)_{\dot\alpha\beta} = (\sigma^a)_{\beta\dot\alpha} \, , \qquad \Big((\sigma^a)^{\alpha\dot\beta}\Big)^\dagger = (\bar\sigma^a)^{\dot\alpha\beta} = (\sigma^a)^{\beta\dot\alpha} \, . \tag{244}$$

The van der Waerden symbols obey the completeness relations

$$\begin{aligned} (\sigma^a)_{\alpha\dot\alpha}(\bar\sigma_a)^{\dot\beta\beta} &= -2\,\delta^\beta_\alpha\,\delta^{\dot\beta}_{\dot\alpha} \, , \\ (\sigma^a)_\alpha{}^{\dot\alpha}(\bar\sigma^b)_{\dot\alpha}{}^\beta &= \eta^{ab}\delta^\beta_\alpha + (\sigma^{ab})_\alpha{}^\beta \, , \\ (\bar\sigma^a)_{\dot\alpha}{}^\alpha(\sigma^b)_\alpha{}^{\dot\beta} &= \eta^{ab}\delta^{\dot\beta}_{\dot\alpha} + (\bar\sigma^{ab})_{\dot\alpha}{}^{\dot\beta} \, , \end{aligned} \tag{245}$$

where $(\sigma^{ab})_{\alpha\beta} = -(\sigma^{ba})_{\alpha\beta} = (\sigma^{ab})_{\beta\alpha}$ and $(\bar\sigma^{ab})_{\dot\alpha\dot\beta} = -(\bar\sigma^{ba})_{\dot\alpha\dot\beta} = (\bar\sigma^{ab})_{\dot\beta\dot\alpha}$. These quantities obey the decomposition rules

$$\begin{aligned} (\sigma^{ab})_\alpha{}^\gamma(\sigma^c)_{\gamma\dot\beta} &= i\epsilon^{abcd}(\sigma_d)_{\alpha\dot\beta} + 2\,i\,(\sigma^{[a})_{\alpha\dot\beta}\,\eta^{b]c} \, , \\ (\sigma^{ab})_\alpha{}^\gamma(\sigma^{cd})_\gamma{}^\beta &= \Big(i\,\epsilon^{abcd} + 2\eta^{b[c}\eta^{d]a}\Big)\,\delta^\beta_\alpha + 4(\sigma^{[a|[d})_\alpha{}^\beta\,\eta^{c]|b]} \, , \end{aligned} \tag{246}$$

and they have the duality properties

$$\tfrac{1}{2}\epsilon_{abcd}(\sigma^{cd})_{\alpha\beta} = i(\sigma_{ab})_{\alpha\beta} \, , \qquad \tfrac{1}{2}\epsilon_{abcd}(\bar\sigma^{cd})_{\dot\alpha\dot\beta} = -i(\bar\sigma_{ab})_{\dot\alpha\dot\beta} \, , \tag{247}$$

where the tensorial ϵ-symbol is defined by $\epsilon^{0123} = -\epsilon_{0123} = 1$.

For $SO(3,2)$ we take $\eta_{AB} = \mathrm{diag}(-+++-)$ $(A, B = 0,1,2,3,5)$ and work with four-component, Majorana spinors $\Psi_{\underline\alpha}$ and Γ-matrices

$$(\Gamma^A)_{\underline\alpha}{}^{\underline\beta} = \begin{cases} i(\gamma^a\gamma^5)_{\underline\alpha}{}^{\underline\beta} & \text{for } A = a \\ i(\gamma^5)_{\underline\alpha}{}^{\underline\beta} & \text{for } A = 5 \end{cases} \, , \tag{248}$$

where γ^a are symmetric $SO(3,1)$ γ-matrices and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ (such that $(\gamma^5)^2 = 1$). In the Dirac representation we choose

$$C_{\underline{\alpha\beta}} = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} ,$$

$$(\gamma^a)_{\underline{\alpha}}{}^{\underline{\beta}} = \begin{pmatrix} 0 & (\sigma^a)_{\alpha\dot{\beta}} \\ -(\bar{\sigma}^a)^{\dot{\alpha}\beta} & 0 \end{pmatrix} , \qquad (\gamma^5)_{\underline{\alpha}}{}^{\underline{\beta}} = \begin{pmatrix} -\delta_\alpha^\beta & 0 \\ 0 & \delta^{\dot{\alpha}}_{\dot{\beta}} \end{pmatrix} . \tag{249}$$

An $SO(3,1)$ Weyl spinor ψ_α and its hermitian conjugate $(\psi_\alpha)^\dagger = \bar{\psi}_{\dot{\alpha}}$ can be used to represent a real Majorana spinor $\Psi^{(+)}_{\underline{\alpha}}$ as well as a purely imaginary Majorana spinor $\Psi^{(-)}_{\underline{\alpha}}$:

$$\Psi^{(\pm)}_{\underline{\alpha}} = \begin{pmatrix} \psi_\alpha \\ \pm\bar{\psi}^{\dot{\alpha}} \end{pmatrix} , \qquad \overline{\Psi}^{(\pm)} := (\Psi^{(\pm)})^\dagger\gamma^0 = \pm(\Psi^{(\pm)})^T C . \tag{250}$$

B The $OSp(8|4)$ Subalgebra of $shs^E(8|4)$

In the expansion (19), we find the quadratic, homogeneous polynomial

$$P_2(Y,\theta) := \frac{1}{4i}\Big(\lambda^{ij}\theta_{ij} + 2\,\epsilon_{i\,\alpha}\,y^\alpha\theta^i + 2\,\bar{\epsilon}_{i\,\dot{\alpha}}\,\bar{y}^{\dot{\alpha}}\theta^i + \ell_{\alpha\beta}\,y^\alpha y^\beta + 2\,a_{\alpha\dot{\beta}}\,y^\alpha\bar{y}^{\dot{\beta}} + \ell_{\dot{\alpha}\dot{\beta}}\,\bar{y}^{\dot{\alpha}}\bar{y}^{\dot{\beta}}\Big) , \tag{251}$$

where $\lambda^{ij}, \epsilon_{i\alpha}, \ell_{\alpha\beta}, a_{\alpha\dot{\beta}}$ are the parameters for $SO(8)$, supersymmetry, Lorentz transformations and translations respectively. The corresponding generators

$$\begin{aligned} Q_{\alpha i} &= \tfrac{1}{2}y_\alpha\theta_i , & Q_{\dot{\alpha} i} &= \tfrac{1}{2}\bar{y}_{\dot{\alpha}}\theta_i , & T^{ij} &= \tfrac{i}{2}\theta^{ij} \\ M_{\alpha\beta} &= \tfrac{1}{2}y_\alpha y_\beta , & P_{\alpha\dot{\beta}} &= \tfrac{1}{2}y_\alpha\bar{y}_{\dot{\beta}} , & M_{\dot{\alpha}\dot{\beta}} &= \tfrac{1}{2}\bar{y}_{\dot{\alpha}}\bar{y}_{\dot{\beta}} , \end{aligned} \tag{252}$$

obey the $D=4$, $N=8$ anti de Sitter superalgebra $OSp(8|4)$

$$\begin{aligned} \{\,Q_{\alpha i}\,,\,Q_{\beta j}\,\}_\star &= \delta_{ij}\,M_{\alpha\beta} + \epsilon_{\alpha\beta}\,T_{ij} , & \{\,Q_{\alpha i}\,,\,Q_{\dot{\beta} j}\,\}_\star &= \delta_{ij}\,P_{\alpha\dot{\beta}} , \\ [\,T_{ij}\,,\,Q_{\alpha k}\,]_\star &= i\,\delta_{jk}\,Q_{\alpha i} - i\,\delta_{ik}\,Q_{\alpha j} , & [\,T_{ij}\,,\,T_{kl}\,]_\star &= i\,\delta_{jk}\,T_{il} + 3 \text{ terms} , \\ [\,M_{\alpha\beta}\,,\,Q_{\gamma i}\,]_\star &= i\,\epsilon_{\alpha\gamma}\,Q_{\beta i} + i\,\epsilon_{\beta\gamma}\,Q_{\alpha i} , & [\,P_{\alpha\dot{\beta}}\,,\,Q_{\gamma i}]_\star &= i\,\epsilon_{\alpha\gamma}\,Q_{\dot{\beta} i} , \\ [\,M_{\alpha\beta}\,,\,M_{\gamma\delta}\,]_\star &= i\,\epsilon_{\alpha\gamma}\,M_{\beta\delta} + 3 \text{ terms} , & [\,M_{\alpha\beta},\,,P_{\gamma\dot{\delta}}\,]_\star &= i\,\epsilon_{\alpha\gamma}\,P_{\beta\dot{\delta}} + i\,\epsilon_{\beta\gamma}\,P_{\alpha\dot{\delta}} , \\ [\,P_{\alpha\dot{\beta}}\,,\,P_{\gamma\dot{\delta}}\,]_\star &= i\,\epsilon_{\dot{\beta}\dot{\delta}}\,M_{\alpha\gamma} + i\,\epsilon_{\alpha\gamma}\,M_{\dot{\beta}\dot{\delta}} , \end{aligned} \tag{253}$$

and hermitian conjugates. The following change of basis for the $SO(3,2)$ subalgebra

$$M_{ab} = \tfrac{1}{4}(\sigma_{ab})^{\alpha\beta}M_{\alpha\beta} + \tfrac{1}{4}(\bar{\sigma}_{ab})^{\dot{\alpha}\dot{\beta}}M_{\dot{\alpha}\dot{\beta}} , \qquad M_{a4} = \tfrac{1}{2}(\sigma_a)^{\alpha\dot{\beta}}M_{\alpha\dot{\beta}} , \qquad a,b = 0,1,...,3 , \tag{254}$$

yields

$$[\, M_{AB} \,,\, M_{CD} \,]_\star \;=\; -i\eta_{BC} M_{AD} + 3 \text{ terms} \;, \qquad A,B = 0,1,...,4 \;. \tag{255}$$

If we let Ω to be the $SO(3,2)$ connection one-form given in the two $SO(3,2)$ bases by

$$\Omega \;=\; \tfrac{1}{4i}\left(\omega_{\alpha\beta} y^\alpha y^\beta + \bar\omega_{\dot\alpha\dot\beta} \bar y^{\dot\alpha} \bar y^{\dot\beta} + 2\omega_{\alpha\dot\alpha} y^\alpha \bar y^{\dot\alpha}\right) \;=\; \tfrac{1}{2i}\omega_{AB} M^{AB} \;, \tag{256}$$

then we find the following relation between the components gauge fields

$$\omega_{\alpha\beta} \;=\; \tfrac{1}{4}(\sigma^{ab})_{\alpha\beta}\omega_{ab} \;, \qquad \omega_{\dot\alpha\dot\beta} \;=\; \tfrac{1}{4}(\bar\sigma^{ab})_{\dot\alpha\dot\beta}\omega_{ab} \;, \qquad \omega_{\alpha\dot\alpha} \;=\; -\tfrac{1}{2}(\sigma^a)_{\alpha\dot\alpha}\omega_a \;, \tag{257}$$

where $\omega_a := \omega_{a4}$. One may notice that

$$y_{\alpha_1\cdots\alpha_{4k+2-n}}\theta_{i_1\cdots i_n} \;=\; M_{(\alpha_1\alpha_2}\star\cdots\star M_{\alpha_{4k+1-2m}\alpha_{4k+2-2m}}\star Q_{\alpha_{4k+3-2m}[i_1}\star\cdots\star Q_{\alpha_{4k+2-m})i_m]} \;. \tag{258}$$

Hence, considered as a vector space, $shs^E(8|4)$ can be identified with the subspace of the $OSp(8|4)$ enveloping algebra which is spanned by odd, fully (anti-)symmetrized functions. However, when considered as algebras, $shs^E(8|4)$ and the $OSp(8|4)$ enveloping algebra differ from each other. Actually, the $OSp(8|4)$ relations (253) in combination with (258) do not suffice to determine the commutator of two general elements in $shs^E(8|4)$. Moreover, a representation of $shs^E(8|4)$ does not need to represent the relation (258), as is the case, for instance , with the tensor product representation (265).

C Oscillator Realization of $shs^E(8|4)$

One way to obtain unitary representations of $shs^E(8|4)$ is consider tensor products of the Fock space Φ obtained by acting on a ground state $|0\rangle$ with a pair of bosonic creation operators $\hat a^\dagger_p$ ($p = 1,2$) and fermionic creation operators $\hat\psi_A$ ($A = 1,...,4$) obeying [36, 37]

$$\begin{aligned}
[\, \hat a_p \,,\, \hat a^\dagger_q \,] \;&=\; \delta_{pq} \;, \qquad p,q = 1,2 \;, \\
\{\, \hat\psi_A \,,\, \hat\psi^\dagger_B \,\} \;&=\; \delta_{AB} \;, \qquad A,B = 1,2,3,4 \\
\hat a_p\, |0\rangle \;&=\; \hat\psi_A\, |0\rangle \;=\; 0 \;.
\end{aligned} \tag{259}$$

The representation on Φ of the element F of $\mathcal{A}$ given in (3) is given by

$$\hat F \;=\; F(\hat y, \hat{\bar y}; \hat\theta) \;, \tag{260}$$

where $\hat y_\alpha$, $\hat{\bar y}_{\dot\alpha}$ and $\hat\theta^i$ are given by

$$\hat a_1 \;=\; \tfrac{1}{2}(\hat y_1 + i\hat{\bar y}_{\dot 2}) \;, \qquad \hat a_2 \;=\; \tfrac{1}{2}(\hat{\bar y}_{\dot 1} + i\hat y_2) \;, \qquad \hat\psi_A \;=\; \tfrac{1}{2}(\hat\theta_{2A-1} + i\hat\theta_{2A}) \;. \tag{261}$$

Notice that the operator $\hat{F}$ is fully (anti)symmetrized, or Weyl ordered. The $\star$ product of elements in $\mathcal{A}$ given in (6-8) is then represented in Φ by the ordinary operator product:

$$(\widehat{F\star G}) \;=\; \hat{F}\hat{G}\;. \tag{262}$$

The resulting unitary representation of $shs^E(8|4)$ acts reducible on Φ, since the elements of $shs^E(8|4)$ are even polynomials, and as a result Φ actually splits into two UIR's of $shs^E(8|4)$, namely

$$\Phi \;=\; \Phi_e \oplus \Phi_o\;, \tag{263}$$

where the states in Φ_e (Φ_o) are made up by acting on $|0\rangle$ with an even (odd) total number of fermionic and bosonic oscillators. The two spaces Φ_e and Φ_o remain irreducible under the $OSp(8|4)$ subalgebra (253) of $shs^E(8|4)$ and, using the notation introduced in (29), they are labeled as follows

$$\Phi_e \;=\; [D(\tfrac{1}{2},0)\otimes 8_s]\oplus[D(1,\tfrac{1}{2})\otimes 8_c]\;,\qquad \Phi_o \;=\; [D(\tfrac{1}{2},0)\otimes 8_c]\oplus[D(1,\tfrac{1}{2})\otimes 8_s]\;, \tag{264}$$

where 8_s (8_c) are the two 8-dimensional subspaces of the 16-dimensional Fock space of the fermionic oscillators, obtained by acting with an even (odd) number of fermionic creation operators on the vacuum. The representation of the element P in $shs^E(8|4)$ on the tensor product $\Phi\otimes\Phi$ is defined by

$$\hat{P}(|u\rangle\otimes|v\rangle) \;:=\; (\hat{P}\,|u\rangle)\otimes|v\rangle + (-1)^{uP}\,|u\rangle\otimes(\hat{P}\,|v\rangle)\;, \tag{265}$$

where u and P in the exponents denote Grassmann parities (the vacuum is taken to be even). Eq. (265) implies that the tensor product is *not* a representation of the $\star$ algebra $\mathcal{A}$. The tensor product form a reducible, unitary representation of $shs^E(8|4)$, containing the invariant subspaces $[\Phi_\lambda\otimes\Phi_{\lambda'}]_{S,A}$, where $\lambda,\lambda' = e,o$ (see (263)), obtained by symmetrization (S) and anti-symmetrization (A) of the tensor product according to the rule

$$\begin{aligned}
\Big[\,|u\rangle\otimes|v\rangle\,\Big]_S &= |u\rangle\otimes|v\rangle + (-1)^{uv}\,|v\rangle\otimes|u\rangle\;,\\
\Big[\,|u\rangle\otimes|v\rangle\,\Big]_A &= |u\rangle\otimes|v\rangle - (-1)^{uv}\,|v\rangle\otimes|u\rangle\;.
\end{aligned} \tag{266}$$

The result in Table 1 for the $SO(3,2)\times SO(8)$ content of $[\Phi_e\otimes\Phi_e]_S$ is derived using the following decomposition rules for the $SO(3,2)$ content

$$\begin{aligned}
\Big[D(\tfrac{1}{2},0)\otimes D(\tfrac{1}{2},0)\Big]_S &= D(1,0)\oplus D(3,2)\oplus D(5,4)\oplus D(7,6)\oplus\cdots\;,\\
\Big[D(\tfrac{1}{2},0)\otimes D(\tfrac{1}{2},0)\Big]_A &= D(2,1)\oplus D(4,3)\oplus D(6,5)\oplus\cdots\;,\\
\Big[D(1,\tfrac{1}{2})\otimes D(1,\tfrac{1}{2})\Big]_S &= D(2,1)\oplus D(4,3)\oplus D(6,5)\oplus\cdots\;,\\
\Big[D(1,\tfrac{1}{2})\otimes D(1,\tfrac{1}{2})\Big]_A &= D(2,0)\oplus D(3,2)\oplus D(5,4)\oplus D(7,6)\oplus\cdots\;,\\
\Big[D(\tfrac{1}{2},0)\otimes D(1,\tfrac{1}{2})\Big]_{S,A} &= D(\tfrac{3}{2},\tfrac{1}{2})\oplus D(\tfrac{5}{2},\tfrac{3}{2})\oplus D(\tfrac{7}{2},\tfrac{5}{2})\oplus\cdots\;,
\end{aligned} \tag{267}$$

and the $SO(8)$ content

$$
\begin{aligned}
8_s \otimes 8_s &= 1_S + 28_A + 35_S^+ \ , \\
8_s \otimes 8_c &= 8_c \otimes 8_s \;=\; 8_v + 56 \\
8_c \otimes 8_c &= 1_A + 28_S + 35_A^- \ ,
\end{aligned} \tag{268}
$$

where the odd Grassmann parity of the states in 8_c has been taken into account in the last equation.

D Symplectic Differentiation and Integration Formula

Using (72), one finds the following contraction rules

$$
\begin{aligned}
y_\alpha \star F(Z,Y) &= y_\alpha F(Z,Y) + \left[-i\frac{\partial}{\partial z^\alpha} + i\frac{\partial}{\partial y^\alpha}\right] F(Z,Y) \ , \\
z_\alpha \star F(Z,Y) &= z_\alpha F(Z,Y) + \left[-i\frac{\partial}{\partial z^\alpha} + i\frac{\partial}{\partial y^\alpha}\right] F(Z,Y) \ , \\
F(Z,Y) \star y_\alpha &= y_\alpha F(Z,Y) + \left[-i\frac{\partial}{\partial z^\alpha} - i\frac{\partial}{\partial y^\alpha}\right] F(Z,Y) \ , \\
F(Z,Y) \star z_\alpha &= z_\alpha F(Z,Y) + \left[i\frac{\partial}{\partial z^\alpha} + i\frac{\partial}{\partial y^\alpha}\right] F(Z,Y) \ , \\
\\
\bar{y}_{\dot\alpha} \star F(Z,Y) &= \bar{y}_{\dot\alpha} F(Z,Y) + \left[i\frac{\partial}{\partial \bar{z}^{\dot\alpha}} + i\frac{\partial}{\partial \bar{y}^{\dot\alpha}}\right] F(Z,Y) \ , \\
\bar{z}_{\dot\alpha} \star F(Z,Y) &= \bar{z}_{\dot\alpha} F(Z,Y) + \left[-i\frac{\partial}{\partial \bar{z}^{\dot\alpha}} - i\frac{\partial}{\partial \bar{y}^{\dot\alpha}}\right] F(Z,Y) \ , \\
F(Z,Y) \star \bar{y}_{\dot\alpha} &= \bar{y}_{\dot\alpha} F(Z,Y) + \left[i\frac{\partial}{\partial \bar{z}^{\dot\alpha}} - i\frac{\partial}{\partial \bar{y}^{\dot\alpha}}\right] F(Z,Y) \ , \\
F(Z,Y) \star \bar{z}_{\dot\alpha} &= \bar{z}_{\dot\alpha} F(Z,Y) + \left[i\frac{\partial}{\partial \bar{z}^{\dot\alpha}} - i\frac{\partial}{\partial \bar{y}^{\dot\alpha}}\right] F(Z,Y) \ .
\end{aligned} \tag{269}
$$

where $F(Z,Y)$ is an arbitrary function. The linear differential equations in Z-space of the type

$$
\partial f \;=\; g \;=\; dz^\alpha g_\alpha + d\bar{z}^{\dot\alpha} g_{\dot\alpha} \ , \tag{270}
$$

$$
\partial(dz^\alpha f_\alpha + d\bar{z}^{\dot\alpha} f_{\dot\alpha}) \;=\; h \;=\; \tfrac{1}{2}dz^2 h + \tfrac{1}{2}d\bar{z}^2\tilde{h} + dz^\alpha \wedge d\bar{z}^{\dot\alpha} h_{\alpha\dot\alpha} \ , \tag{271}
$$

encountered in the perturbative expansion of the higher spin equations around the anti de Sitter vacuum, have the solutions [28]

$$
f(z,\bar{z}) \;=\; f(0,0) + \int_0^1 dt \left[\, z^\alpha g_\alpha(tz,t\bar{z}) + \bar{z}^{\dot\alpha} g_{\dot\alpha}(tz,t\bar{z})\right] \ , \tag{272}
$$

$$f_\alpha(z,\bar z) \;=\; \frac{\partial}{\partial z^\alpha}k(z,\bar z) - \int_0^1 dt\, t\left[\, z_\alpha\, h(tz,t\bar z) + \bar z^{\dot\alpha} h_{\alpha\dot\alpha}(tz,t\bar z)\right]\ , \qquad (273)$$

$$f_{\dot\alpha}(z,\bar z) \;=\; \frac{\partial}{\partial \bar z^{\dot\alpha}}k(z,\bar z) - \int_0^1 dt\, t\left[\, \bar z_{\dot\alpha}\, \tilde h(tz,t\bar z) - z^{\alpha} h_{\alpha\dot\alpha}(tz,t\bar z)\right]\ , \qquad (274)$$

where $f(0)$ is an arbitrary constant and $k(z,\bar z)$ an arbitrary function. In proving these formula one makes use of

$$t\frac{d}{dt}h(tz) \;=\; z^\alpha \frac{\partial}{\partial z^\alpha}h(zt)\ . \qquad (275)$$

Therefore, to apply them correctly to (146-148), one must expand the $\star$ products in the right hand sides of (146-148) *before* one replaces z and $\bar z$ by tz and $t\bar z$ as indicated in (272-274). This is so because the contractions alter the functional dependence on z and $\bar z$ such that if one would replace z and $\bar z$ by tz and $t\bar z$ before one expands the $\star$ products then (275) is no longer valid. As a matter of fact, if we let $A := A(tz,t\bar z,y,\bar y)$ and $B := B(tz,t\bar z,y,\bar y)$ then we have

$$t\frac{d}{dt}(A\star B) \;=\; \left[z^\alpha\frac{\partial}{\partial z^\alpha} + \bar z^{\dot\alpha}\frac{\partial}{\partial \bar z^{\dot\alpha}}\right](A\star B) - 2i\,\epsilon^{\alpha\beta}\frac{\partial}{\partial z^\alpha}A\star\frac{\partial}{\partial z^\beta}B - 2i\,\epsilon^{\dot\alpha\dot\beta}\frac{\partial}{\partial \bar z^{\dot\alpha}}A\star\frac{\partial}{\partial \bar z^{\dot\beta}}B\ . \qquad (276)$$

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